EECS 555 Problem Set 1

Due: Wednesday, January 20. Reading: Chapter 1, 2 and Appendix A.

1. Prove $P\{\cup_{i=1}^{M} A_i\} \leq \sum_{i=1}^{M} P\{A_i\}$.

2. Let H_0 and H_1 be two events. Let $p_i(x_1, \ldots, x_n)$ be the conditional density of $X_1, \ldots, X_n$ given H_i occurred.

   (a) Assume that given H_i, $\{X_j\}_{j=1}^{n}$ is a sequence of i.i.d. Gaussian random variables with mean $(-1)^i\sqrt{E}$ and variance $N_0/2$. Use the Chernoff bound to show that

$$P\left\{p_1(\underline{X}) \geq p_0(\underline{X})|H_0\right\} \leq e^{-nE/N_0}$$

   (b) Now let $X_1, \ldots, X_n$ be independent discrete random variables taking values $+1$ and -1 with

$$\begin{aligned} P\{X_i = +1|H_0\} &= p, \\ P\{X_i = -1|H_0\} &= 1-p, \\ P\{X_i = -1|H_1\} &= p, \\ P\{X_i = +1|H_1\} &= 1-p. \end{aligned}$$

   Thus if the number of components of $\underline{x}$ equal to $+1$ is d then $p_0(\underline{x}) = p^d(1-p)^{n-d}$ and $p_1(\underline{x}) = p^{n-d}(1-p)^d$. Use the Chernoff bound to show that

$$P\left\{p_1(\underline{X}) \geq p_0(\underline{X})|H_0\right\} \leq e^{-n(-\ln\sqrt{4p(1-p)})}$$

   (c) Again let $X_1, \ldots, X_n$ be independent discrete random variables with X_i taking nonnegative integer values only. Let

$$p_0(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{\lambda_0^{x_i} e^{-\lambda_0}}{x_i!}, \quad x_i \geq 0, \quad 1 \leq i \leq n$$

   and

$$p_1(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!}$$

   If $\lambda_1 > \lambda_0$ find the best Chernoff bound on

$$P_{e,0} = P\left\{p_1(\underline{X}) \geq p_0(\underline{X})|H_0\right\}$$

   and

$$P_{e,1} = P\left\{p_0(\underline{X}) \geq p_1(\underline{X})|H_1\right\}.$$

   Let s_i^* be the optimal value of s for minimizing the bound to $P_{e,i}$. Show $s_1^* = 1 - s_0^*$.

   (d) Again let $X_1, \ldots, X_{2n}$ be independent discrete random variables with X_i taking nonnegative integer values only. Let

$$p_0(x_1, \ldots, x_{2n}) = \prod_{i=1}^{n} \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!} \prod_{i=n+1}^{2n} \frac{\lambda_0^{x_i} e^{-\lambda_0}}{x_i!}, \quad x_i \geq 0, \quad 1 \leq i \leq 2n$$

and

$$p_1(x_1, \ldots, x_{2n}) = \prod_{i=1}^{n} \frac{\lambda_0^{x_i} e^{-\lambda_0}}{x_i!} \prod_{i=n+1}^{2n} \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!}$$

If $\lambda_1 > \lambda_0$ find the best Chernoff bound on

$$P_{e,0} = P\left\{p_1(\underline{X}) \geq p_0(\underline{X}) | H_0\right\}$$

and

$$P_{e,1} = P\left\{p_0(\underline{X}) \geq p_1(\underline{X}) | H_1\right\}.$$

3. (a) Consider any two binary signals $s_0(t), s_1(t)$ with $E_i = \| s_i(t) \|^2$ the energy of signal $s_i(t)$ and

$$\begin{aligned} \rho &= (s_0, s_1)/\bar{E} \\ &= \int s_0(t) s_1(t) dt / \bar{E} \end{aligned}$$

where $\bar{E} = (E_0 + E_1)/2$. Assume that signal $s_0(t)$ is transmitted with probability π_0 and signal $s_1(t)$ is transmitted with probability π_1 with $\pi_0 + \pi_1 = 1$. The received signal is the transmitted signal plus additive white Gaussian noise with two-sided power spectral density $N_0/2$.

(a) Find the optimum receiver. (Do **not** assume that $\pi_0 = \pi_1$).

(b) Find the average error probability of the optimum receiver.

(c) If now $\pi_0 = \pi_1$ find the value of ρ that minimizes the error probability. (Note that by Schwarz' inequality $-1 \leq \rho \leq 1$).

4. Find the test (decision rule) that minimizes average error probability for the following detection theory problem. Assume $\sigma_1 > \sigma_0$ and $\pi_0 = \pi_1 = 1/2$.

$$H_0 : \; X \quad p_0(x) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-x^2/(2\sigma_0^2)\right\}$$

$$H_1 : \; X \quad p_1(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-x^2/(2\sigma_1^2)\right\}$$

5. One of two equally likely messages is to be transmitted over an AWGN channel by means of the two signals

$$s_0(t) = \sqrt{2E_s/T} p_T(t) \cos \omega_c t$$

$$s_1(t) = \sqrt{2E_s/T} p_T(t) \cos(\omega_c t + \phi)$$

where $p_T(t)$ is a unit amplitude pulse of duration T.

(a) Find the optimum receiver. (You may assume that $\omega_c T$ is an integer multiple of π.)

(b) Calculate the error probability in terms of the Q function.