Chapter 3

Error Probability for M Signals

In this chapter we discuss the error probability in deciding which of M signals was transmitted over an arbitrary channel. We assume the signals are represented by a set of N orthonormal functions. Writing down an expression for the error probability in terms of an N-dimensional integral is straightforward. However, evaluating the integrals involved in the expression in all but a few special cases is very difficult or impossible if N is fairly large (e.g. $N > 4$). For the special case of orthogonal signals in the last chapter we derived the error probability as a single integral. Because of the difficulty of evaluating the error probability in general bounds are needed to determine the performance. Different bounds have different complexity of evaluation. This first bound we derive is known as the Gallager bound. We apply this bound to the case of orthogonal signals (for which the true answer is already known). The Gallager bound has the property that when the number of signals become large the bound becomes tight. However, the bound is fairly difficult to evaluate for many signal sets. A special case of the Gallager bound is the Union-Bhattacharayya bound. This is simpler than the Gallager bound to evaluate but also is looser than the Gallager bound. The last bound considered is the union bound. This bound is tighter than the Union-Bhattacharayya bound and the Gallager bound for sufficiently high signal-to-noise ratios. Finally we consider a simple random coding bound on the ensemble of all signal sets using the Union-Bhattacharayya bound.

The general signal set we consider has the form

$$s_i(t) = \sum_{j=0}^{N-1} s_{i,j}\varphi_j(t), i = 0, 1, ..., M-1.$$

The optimum receiver does a correlation with the N orthonormal waveforms to form the decision variables.

$$r_j = \int_0^T r(t)\varphi_j(t)dt, j = 0, 1, ..., N-1.$$

The decision regions are for equally likely signals given by

$$R_i \quad \stackrel{\Delta}{=} \quad \{\mathbf{r} : p_i(\mathbf{r}) > p_j(\mathbf{r}), \forall j \neq i\}$$

The error probability is then determined by

$$P_{e,i} \quad = \quad P(\bigcup_{j=0, j\neq i}^{M-1} R_j | H_i).$$

For all but a few small dimensional signals or signals with special structures (such as orthogonal signal sets) the exact error probability is very difficult to calculate.

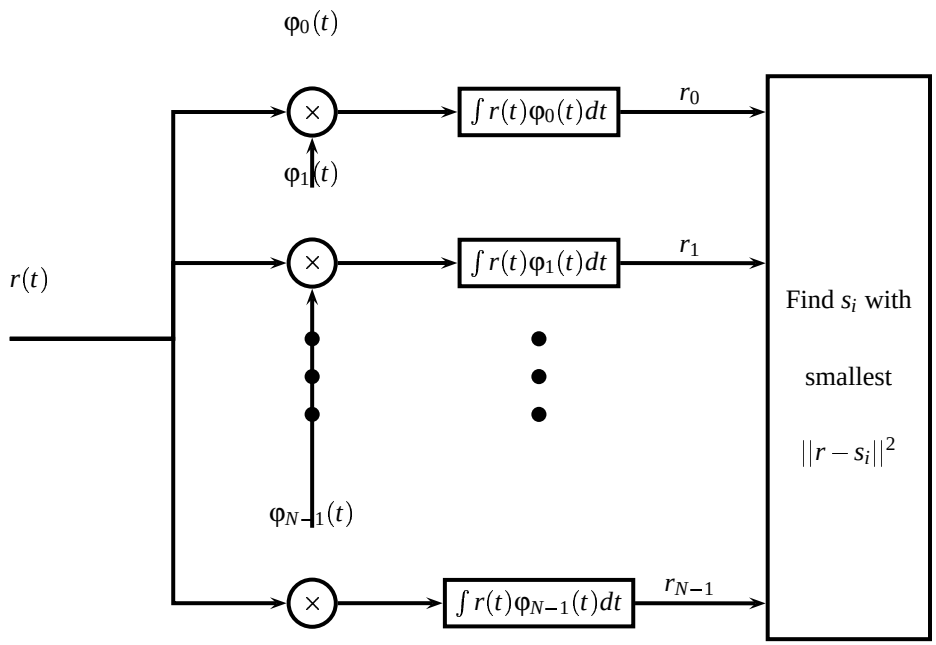


Figure 3.1: Optimum Receiver in Additive White Gaussian Noise

1. Error Probability for Orthogonal Signals

Represent the M signals in terms of M orthogonal function $\varphi_i(t)$ as follows

$$\begin{aligned} s_0(t) &= \sqrt{E}\varphi_0(t) \\ s_1(t) &= \sqrt{E}\varphi_1(t) \\ . & \quad . \\ . & \quad . \\ . & \quad . \\ s_{M-1}(t) &= \sqrt{E}\varphi_{M-1}(t). \end{aligned}$$

As shown in the previous chapter we need to find the largest value of $(r(t), s_j(t))$ for $j = 0, ..., M-1$. Instead we will normalize this and determine the largest value of $r_j = (r(t), s_j(t))/\sqrt{E}$. To determine the error probability we need to determine the statistics of r_j. Assume signal s_i is transmitted. Then

$$r_j \triangleq \int_0^T r(t)\varphi_j(t)dt$$

$$\begin{aligned} E[r_j|H_i] &= E[\int_0^T r(t)\varphi_j(t)dt|H_i] \\ &= \int_0^T E[r(t)|H_i]\varphi_j(t)dt \\ &= \int_0^T E[s_i(t)+n(t)]\varphi_j(t)dt \\ &= \int_0^T \sqrt{E}\varphi_i(t)\varphi_j(t)dt \\ &= \sqrt{E}\delta_{i,j}. \end{aligned}$$

The variance of r_j is determined as follows. Given H_i

$$\begin{aligned}
r_j - E[r_j|H_i] &= \int_0^T n(t)\varphi_j(t)dt \\
E[(r_j - E[r_j|H_i])^2|H_i] &= \int_0^T \int_0^T E[n(t)n(s)]\varphi_j(t)\varphi_j(s)dtds \\
&= \int_0^T \int_0^T K(t,s)\varphi_j(t)\varphi_j(s)dtds \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\varphi_j(t)\varphi_j(s)dtds \\
&= \int_0^T \frac{N_0}{2}\varphi_j(t)\varphi_j(t)dt \\
&= \frac{N_0}{2}.
\end{aligned}$$

Furthermore, each of these random variables is Gaussian (and independent).

$$P(\text{error}) = 1 - P(\text{correct})$$

$$P(\text{correct}) = \sum_{i=0}^{M-1} P(H_i|H_i)\pi_i$$

$$\begin{aligned}
P_{c,i} \triangleq P(H_i|H_i) &= P(r_i > r_j,\ \forall j \neq i|H_i) \\
&= E[P(r_i > r_j,\ \forall j \neq i|H_i, r_i)] \\
&= E[\prod_{j=0, j\neq i}^{M-1} P(r_i > r_j|H_i, r_i)] \\
&= E[\prod_{j=0, j\neq i}^{M-1} \Phi(\frac{r_i}{\sqrt{N_0/2}})] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\{-\frac{1}{N_0}(r_i - \sqrt{E})^2\}\Phi^{M-1}(\frac{r_i}{\sqrt{N_0/2}})dr_i.
\end{aligned}$$

$\left(\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}du\right)$. Now let $u = \frac{r_i}{\sqrt{N_0/2}}$. Then

$$\begin{aligned}
P_{e,i} = 1 - P_{c,i} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \sqrt{\frac{N_0}{2}} \exp\{-(\sqrt{\frac{N_0}{2}}u - \sqrt{E})^2/N_0\}[1 - \Phi^{M-1}(u)]du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-(u - \sqrt{\frac{2E}{N_0}})^2/2\}[1 - \Phi^{M-1}(u)]du \\
&= (M-1)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \Phi(u - \sqrt{\frac{2E}{N_0}})\Phi^{M-2}(u)e^{-u^2/2}du
\end{aligned}$$

where the last step follows from using an integration by parts approach. Later on we will find an upper bound on the above that is more insightful. It is possible to determine (using L'Hospital's rule) the limiting behavior of the error probability as $M \to \infty$.

In general if we have M decision variables for an M-ary hypothesis testing problem that are conditionally independent given the true hypothesis and there is a density (distribution) of the decision variable for the true statistic denoted $f_1(x)$ ($F_1(x)$) and a density and distribution function for the other decision variables ($f_2(x), F_2(x)$) then the probability of correct is

$$P_c = \int_{-\infty}^{\infty} f_1(x)F_2^{M-1}(x)dx$$

The probability of error is

$$\begin{aligned} P_e &= 1 - \int_{-\infty}^{\infty} f_1(x) F_2^{M-1}(x) dx \\ &= (M-1) \int_{-\infty}^{\infty} F_1(x) F_2^{M-2}(x) f_2(x) dx \end{aligned}$$

The last formula is many times easier to compute numerically than the first because the former is the difference between two numbers that are very close (for small error probabilities).

2. Gallager Bound

In this section we derive an upper bound on the error probability for M signals received in some form of noise. Let

$$\begin{aligned} R_i &\triangleq \{\mathbf{r} : p_i(\mathbf{r}) > p_j(\mathbf{r}), \forall j \neq i\} \\ \overline{R}_i &\triangleq \{\mathbf{r} : p_i(\mathbf{r}) \leq p_j(\mathbf{r}), \text{for some} j \neq i\} \\ P_{c,i} &\triangleq P(H_i|H_i) \\ P_{e,i} &\triangleq P(\overline{H}_i|H_i) = P(\overline{R}_i|H_i). \end{aligned}$$

Now

$$\overline{R}_i \triangleq \{\mathbf{r} : \frac{p_j(\mathbf{r})}{p_i(\mathbf{r})} \geq 1, \text{ for some } j \neq i\}.$$

For $\lambda \geq 0$ let

$$\widetilde{R}_i \triangleq \{\mathbf{r} : \sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda} \geq 1\}.$$

Claim: Then $\widetilde{R}_i \supset \overline{R}_i$. **Proof:** If $\mathbf{r} \in \overline{R}_i$ then $\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})} \geq 1$ for some $j \neq i$. Thus for some $j \neq i$, $\left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda} \geq 1$ which implies that

$$\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda} \geq 1$$

and thus $\mathbf{r} \in \widetilde{R}_i$. Thus we have shown that $\widetilde{R}_i \supset \overline{R}_i$. Now we use this to upper bound the error probability.

$$\begin{aligned} P_{e,i} &= P(\overline{R}_i|H_i) \leq P(\widetilde{R}_i|H_i) = \int_{\widetilde{R}_i} p_i(\mathbf{r}) d\mathbf{r} \\ &= \int_{R^M} I[\widetilde{R}_i] p_i(\mathbf{r}) d\mathbf{r} \end{aligned}$$

where

$$I[\widetilde{R}_i] = \begin{cases} 1, & \mathbf{r} \in \widetilde{R}_i \\ 0, & \mathbf{r} \notin \widetilde{R}_i. \end{cases}$$

For $\mathbf{r} \in \widetilde{R}_i$ and $\rho > 0$ we have

$$\left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda}\right)^{\rho} \geq 1.$$

For $\mathbf{r} \notin \widetilde{R}_i$ and $\rho > 0$ we have

$$\left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda}\right)^{\rho} \geq 0.$$

Thus

$$I[\widetilde{R}_i] \leq \left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})} \right]^{\lambda} \right)^{\rho}.$$

Applying this bound to the expression for the error probability we obtain

$$\begin{aligned} P_{e,i} &\leq \int_{R^M} \left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})} \right]^{\lambda} \right)^{\rho} p_i(\mathbf{r}) d\mathbf{r} \\ &= \int_{R^M} [p_i(\mathbf{r})]^{1-\lambda\rho} \left(\sum_{j \neq i} [p_j(\mathbf{r})]^{\lambda} \right)^{\rho} d\mathbf{r} \end{aligned}$$

for $\rho > 0$ and $\lambda > 0$. If we let $\lambda = \frac{1}{1+\rho}$ (this is the value that minimizes the bound, see Gallager problem 5.6) the resulting bound is known as the Gallager bound.

$$P_{e,i} \leq \int_{R^M} [p_i(\mathbf{r})]^{\frac{1}{1+\rho}} \left(\sum_{j \neq i} [p_j(\mathbf{r})]^{\frac{1}{1+\rho}} \right)^{\rho} d\mathbf{r}.$$

If we let $\rho = 1$ we obtain what is known as the Bhattacharayya bound.

$$\begin{aligned} P_{e,i} &\leq \int_{R^M} [p_i(\mathbf{r})]^{\frac{1}{2}} \left(\sum_{j \neq i} [p_j(\mathbf{r})]^{\frac{1}{2}} \right) d\mathbf{r} \\ &= \sum_{j \neq i} \int_{R^M} \sqrt{p_i(\mathbf{r}) p_j(\mathbf{r})} d\mathbf{r}. \end{aligned}$$

The average error probability is then written as

$$P_e = \sum_{i=1}^{M} \pi_i P_{e,i}.$$

1. Example of Gallager bound for M-ary orthogonal signals in AWGN.

$$p_i(\mathbf{r}) = \frac{1}{\sqrt{\pi N_0}} e^{-(r_i - \sqrt{E})^2)/N_0} \prod_{j \neq i} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0}$$

$$\begin{aligned} P_{e,i} &\leq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{\pi N_0}} e^{-(r_i - \sqrt{E})^2/N_0} \prod_{j \neq i} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0} \right]^{\frac{1}{1+\rho}} \\ &\qquad \left\{ \sum_{j \neq i} \left[\frac{1}{\sqrt{\pi N_0}} e^{-(r_j - \sqrt{E})^2/N_0} \prod_{k \neq j} \frac{1}{\sqrt{\pi N_0}} e^{-r_k^2/N_0} \right]^{\frac{1}{1+\rho}} \right\}^{\rho} d\mathbf{r} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\left(\prod_{j=0}^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0} \right) e^{2\sqrt{E} r_i/N_0} e^{-E/N_0} \right]^{\frac{1}{1+\rho}} \\ &\qquad \left\{ \sum_{j \neq i} \left[\left(\prod_{k=0}^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-r_k^2/N_0} \right) e^{2\sqrt{E} r_j/N_0} e^{-E/N_0} \right]^{\frac{1}{1+\rho}} \right\}^{\rho} d\mathbf{r} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\left(\prod_{j=0}^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0} \right) e^{-E/N_0} \right] \\ &\qquad \left[\exp\{ \frac{2\sqrt{E} r_i}{(1+\rho) N_0} \} \right] \left\{ \sum_{j \neq i} \exp\{ \frac{2\sqrt{E} r_j}{(1+\rho) N_0} \} \right\}^{\rho} d\mathbf{r}. \end{aligned}$$

Let

$$g(z) = \exp\{\sqrt{\frac{2E}{N_0}}\frac{z}{1+\rho}\}$$

where $z_i = r_i/\sqrt{N_0/2}$. Then

$$\begin{aligned} P_{e,i} &\leq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=0}^{M-1}\frac{e^{-z_j^2/2}}{\sqrt{2\pi}}e^{-E/N_0}g(z_i)\left[\sum_{j\neq i}g(z_j)\right]^{\rho}d\underline{z} \\ &= e^{-E/N_0}E[g(z_i)]E\left[\sum_{j\neq i}g(z_j)\right]^{\rho}. \end{aligned}$$

Now it is easy to show (by completing the square) that

$$\begin{aligned} E[g(z)] &= \int_{-\infty}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\exp\{\sqrt{\frac{2E}{N_0}}\frac{z}{1+\rho}\}dz \\ &= \exp\left(\frac{E}{N_0(1+\rho)^2}\right). \end{aligned}$$

Let $f(x) = x^{\rho}$ where $0 \leq \rho \leq 1$. Then $f(x)$ is a concave function and thus by Jensen's inequality we have that

$$E[f(X)] \leq f(E[X]).$$

Thus

$$\begin{aligned} E\left(\left[\sum_{j\neq i}g(z_j)\right]^{\rho}\right) &\leq \left(E\left[\sum_{j\neq i}g(z_j)\right]\right)^{\rho} \\ &= \left(\sum_{j\neq i}E[g(z_j)]\right)^{\rho} \\ &= (M-1)^{\rho}(E[g(z_j)])^{\rho}. \end{aligned}$$

Thus

$$\begin{aligned} P_{e,i} &\leq (M-1)^{\rho}e^{-E/N_0}(E[g(z)])^{1+\rho} \\ &= (M-1)^{\rho}\exp\{-\frac{E}{N_0}+(1+\rho)\frac{E}{N_0(1+\rho)^2}\} \\ &\leq \exp\{-\frac{E}{N_0}\left(\frac{\rho}{1+\rho}\right)+\rho\ln M\}. \end{aligned}$$

Now we would like to minimize the bound over the parameter ρ keeping in mind that the bound is only valid for $0 \leq \rho \leq 1$. Let $a = \frac{E}{N_0}$ and $b = \ln M$ and

$$f(\rho) = -a\frac{\rho}{1+\rho}+\rho b.$$

Then

$$f'(\rho) = 0 \Rightarrow \rho = \sqrt{\frac{a}{b}}-1.$$

Since $0 \leq \rho \leq 1$ the minimum occurs at an interior point of the interval $[0,1]$ if

$$1/4 < \frac{\ln M}{\frac{E}{N_0}} < 1$$

in which case the bound becomes

$$P_{e,i} \leq \exp\left\{-\left(\sqrt{\frac{E}{N_0}}-\sqrt{\ln M}\right)^2\right\}.$$

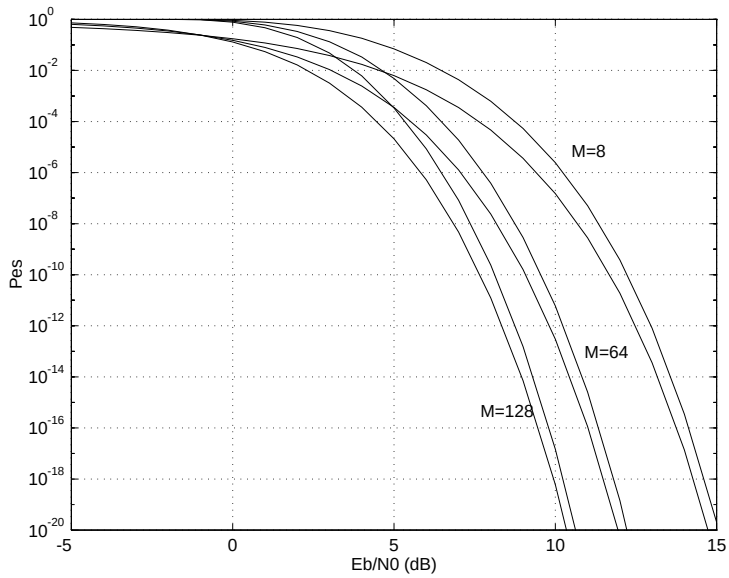


Figure 3.2: Comparison of Gallager Bound and exact error probability for orthogonal signals. In each group the upper curve is the bound and the lower curve is the exact error probability

If $\ln M/(\frac{E}{N_0}) \leq 1/4$ then $\rho_{\min} = 1$ in which case the upper bound becomes $P_{e,i} \leq \exp\{\ln M - \frac{E}{2N_0}\}$. If $\frac{\ln M}{\frac{E}{N_0}} \geq 1$ then $\rho_{\min} = 0$ in which case the upper bound becomes $P_{e,i} \leq 1$. In summary the Gallager bound for M orthogonal signals in white Gaussian noise is

$$P_{e,i} \leq \begin{cases} 1, & \frac{E}{N_0} \leq \ln M \\ \exp\left\{-\left(\sqrt{\frac{E}{N_0}} - \sqrt{\ln M}\right)^2\right\}, & \ln M \leq \frac{E}{N_0} \leq 4\ln M \\ \exp\{-(\frac{E}{2N_0} - \ln M)\}, & \frac{E}{N_0} \geq 4\ln M. \end{cases}$$

Normally a communication engineer is more concerned with the energy transmitted per bit rather than the energy transmitted per signal, E. If we let E_b be the energy transmitted per bit then these are related as follows

$$E_b = \frac{E}{\log_2 M}.$$

Thus the bound on the error probability can be expressed in terms of the energy transmitted per bit as

$$P_{e,i} \leq \begin{cases} 1, & \frac{E_b}{N_0} \leq \ln 2 \\ \exp_2\left\{-\log_2 M\left(\sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2}\right)^2\right\}, & \ln 2 \leq \frac{E_b}{N_0} \leq 4\ln 2 \\ \exp_2\left\{-\log_2 M\left(\frac{E_b}{2N_0} - \ln 2\right)\right\}, & \frac{E_b}{N_0} \geq 4\ln 2 \end{cases}$$

where $\exp_2\{x\}$ denotes 2^x. Note that as $M \to \infty$, $P_e \to 0$ if $\frac{E_b}{N_0} > \ln 2$ = -1.59dB. Below we plot the exact error probability and the Gallager bound for M orthogonal signals for $M = 8, 64, 512$.

3. Bit error probability

So far we have examined the symbol error probability for orthogonal signals. Usually the number of such signals is a power of 2, e.g. 4, 8, 16, 32, ... If so then each transmission of a signal is carrying $\log_2 M$ bits of information.

In this case a communication engineer is usually interested in the bit error probability as opposed to the symbol error probability. Let $d(s_i, s_j)$ be the (Euclidean) distance between s_i and s_j, i.e

$$d^2(s_i, s_j) \triangleq \int_{-\infty}^{\infty} (s_i(t) - s_j(t))^2 dt = \sum_{l=1}^{\infty} |s_{i,l} - s_{j,l}|^2.$$

Now consider any signal set for which the distance between every pair of signals is the same. Orthogonal signal sets with equal energy satisfy this condition. Let $k = \log_2 M$. If s_i is transmitted there are $M-1$ other signals to which an error can be made. The number of signals which cause an error of i bits out of the k is $\binom{k}{i}$. Since all signals are the same distance from s_i the conditional probability of a symbol error causing i bits to be in error is

$$\frac{\binom{k}{i}}{(M-1)}.$$

So the average number of bit error given a symbol error is

$$\sum_{i=1}^{k} i \binom{k}{i} / (M-1) = \frac{k2^{k-1}}{M-1}.$$

So the probability of bit error given symbol error is

$$\frac{1}{k}\frac{k2^{k-1}}{M-1} = \frac{2^{k-1}}{2^k - 1}.$$

So

$$P_{b,i} = \frac{2^{k-1}}{2^k - 1} P_{e,i}$$

and this is true for any equidistant, equienergy signal set.

4. Union Bound

Assume

$$\pi_i = \frac{1}{M}, \quad 0 \le i \le M-1.$$

Let

$$\begin{aligned}
R_i &= \{\underline{r} : p_i(\underline{r}) > p_j(r) \quad \text{for all} \quad j \neq i\}, \\
\overline{R}_i &= \{\underline{r} : p_i(\underline{r}) \le p_j(\underline{r}) \quad \text{for some} \quad j \neq i\}, \\
&= \bigcup_{j=0, j\neq i}^{M-1} \{\underline{r} : p_i(\underline{r}) \le p_j(\underline{r})\}, \\
\overline{R}_{ij} &= \{\underline{r} : p_i(\underline{r}) \le p_j(\underline{r})\}.
\end{aligned}$$

Then

$$\begin{aligned}
P_{e,i} &= P(\underline{r} \in \overline{R}_i \mid H_i) \\
&= P(\underline{r} \in \bigcup_{j\neq i} \overline{R}_{ij} \mid H_i) \\
&\le \sum_{j\neq i} P(\overline{R}_{ij} \mid H_i)
\end{aligned}$$

where

$$P(\overline{R}_{ij} \mid H_i) = P\left\{ \frac{p_i(\underline{r})}{p_j(\underline{r})} \le 1 \mid H_i \right\}$$

This is the union bound.

We now consider the bound for an arbitrary signal set in additive white Gaussian noise.

Let

$$s_i(t) = \sum_{l=1}^{N} s_{il}\varphi_l(t), \quad 0 \le i \le M-1.$$

For additive white Gaussian noise

$$p_i(\underline{r}) = \prod_{l=1}^{N}\left(\frac{\exp\{-\frac{1}{N_0}(r_l - s_{il})^2\}}{\sqrt{\pi N_0}}\right)$$

$$\begin{aligned}\frac{p_i(\underline{r})}{p_j(\underline{r})} &= \prod_{l=1}^{N} \exp\{-\frac{1}{N_0}[(r_l - s_{il})^2 - (r_l - s_{jl})^2]\} \\ &= \exp\left\{\frac{2}{N_0}(\underline{r}, \underline{s}_i - \underline{s}_j) + \frac{E_j - E_i}{N_0}\right\}\end{aligned}$$

where $(\underline{r}, \underline{s}_i - \underline{s}_j) = \sum_{l=1}^{N} r_l(s_{il} - s_{jl})$ and $E_k = \sum_{l=1}^{N} s_{kl}^2$ for $0 \le k \le M-1$. Thus

$$P(\overline{R}_{ij} \mid H_i) = P\left\{\frac{2}{N_0}(\underline{r}, \underline{s}_i - \underline{s}_j) \le \frac{E_i - E_j}{N_0} \mid H_i\right\}.$$

To do this calculation we need to calculate the statistics of the random variable $(r, s_i - s_j)$. The mean and variance are calculated as follows.

$$\begin{aligned}E[(\underline{r}, \underline{s}_i - \underline{s}_j) \mid H_i] &= E[(\underline{n} + \underline{s}_i, \underline{s}_i - \underline{s}_j] \\ &= E_i - (\underline{s}_i, \underline{s}_j). \\ \mathrm{V}ar[(\underline{r}, \underline{s}_i - \underline{s}_j) \mid H_i] &= \mathrm{V}ar\,[(\underline{n} + \underline{s}_i, s_i - s_j)] \\ &= \frac{N_0}{2} \parallel \underline{s}_i - \underline{s}_j \parallel^2 .\end{aligned}$$

Also $(r, \underline{s}_i - \underline{s}_j)$ is a Gaussian random variable. Thus

$$\begin{aligned}P\left\{(\underline{r}, \underline{s}_i - \underline{s}_j) \le \frac{E_i - E_j}{2}\right\} &= \Phi\left(\frac{\frac{E_i - E_j}{2} - (E_i - (s_i, s_j))}{\sqrt{\frac{N_0}{2}}\,\|\underline{s}_i - \underline{s}_j\|}\right) \\ &= Q\left(\frac{E_i - 2(s_i, s_j) + E_j}{\sqrt{2N_0}\,\|\underline{s}_i - \underline{s}_j\|}\right) \\ &= Q\left(\frac{\|\underline{s}_i - \underline{s}_j\|}{\sqrt{2N_0}}\right).\end{aligned}$$

Thus the union bound on the error probability is given as

$$P_{e,i} \le \sum_{j \ne i} Q\left(\frac{\|\underline{s}_j - \underline{s}_i\|}{\sqrt{2N_0}}\right).$$

Note that $\|\underline{s}_i - \underline{s}_j\|^2 = d_E^2(s_i, s_j)$, i.e. the square of the Euclidean distance.

We now use the following to derive the Union-Bhattacharyya bound. This is an alternate way of obtaining this bound. We could have started with the Union-Bhattacharyya bound derived from the Gallager bound, but we would get the same answer.

Fact: $Q(x) \le \frac{1}{2}e^{-x^2/2} \le e^{-x^2/2}, \quad x \ge 0$. (To prove this let X_1 and X_2 be independent Gaussian random variables mean 0 variance 1. Then show $Q^2(x) = P(X_1 \ge x, X_2 \ge x) \le \frac{1}{4}P(X_1^2 + X_2^2 \ge \sqrt{2}\,x)$. Use the fact the $X_1^2 + X_2^2$ has Rayleigh density; see page 29 of Proakis)

Using this fact leads to the bound

$$P_{e,i} \le \sum_{j \ne i} \exp\left\{-\frac{\|\underline{s}_i - \underline{s}_j\|^2}{4N_0}\right\}.$$

This is the Union Bhattacharyya bound for an additive white Gaussian noise channel.

5. Random Coding

Now consider 2^{NM} communication systems corresponding to all possible signals where

$$\begin{aligned} s_{ij} &= \pm\sqrt{E} \\ \|s_i\|^2 &= NE \qquad\qquad 0 \le i \le M-1 \end{aligned}$$

Consider the average error probability, averaged over all possible selections of signal sets

For example: Let $N = 3,\ M = 2 \Rightarrow$. There are $2^{3\times 2} = 2^6 = 64$ possible sets of 2 signals with each signal a linear combination of three orthogonal signals with the coefficients required to be one of two values.

Set number 1	$s_0(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
Set number 2	$s_0(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$
$\vdots$	$\vdots$		
Set number 64	$s_0(t)$	$=$	$+\sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$+\sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$

Let $P_{e,i}(k)$ be the error probability of signal set k given H_i. Then

$$\overline{P}_{e,i} = \frac{1}{2^{NM}} \sum_{k=1}^{2^{NM}} P_{e,i}(k).$$

If $\overline{P}_{e,i} \le \alpha$ then at least one of the of the 2^{NM} signals sets must have $P_{e,i}(k) \le \alpha$ (otherwise $P_{e,i}(k) > \alpha$ for all $k \Rightarrow \overline{P}_{e,i} > \alpha$; contradiction). In other words <u>there exists</u> a signal set with $P_{e,i} \le \alpha$. This is known as the random coding argument. Let $s_{ij},\ 0 \le i \le M-1,\ 1 \le j \le N$ be independent identically distributed random variables with $P(s_{ij} = +\sqrt{E}) = P(s_{ij} = -\sqrt{E}) = \frac{1}{2}$ and $\overline{P}_{e,i} = E[P_{e,i}(s)]$ where the expectation is with respect to the random variables s.

$$\overline{P}_{e,i} = E[P_{e,i}(s)] \le \sum_{j \ne i} E\left[\exp\left\{-\frac{\|s_i - s_j\|^2}{4N_0}\right\}\right].$$

Let $X_{ij} = \|s_i - s_j\|^2 = \sum_{l=1}^{N}(s_{il} - s_{jl})^2$. Then

$$\begin{aligned} P(X_{ij} = 4Em) &= P(s_i \text{ and } s_j \text{ differ in } m \text{ places out of } N) \\ &= \tbinom{N}{m} 2^{-N} \end{aligned}$$

since $P(s_{il} = s_{jl}) = P(s_{il} \ne s_{jl}) = \frac{1}{2}$.

So

$$E[\exp\ \left\{-\frac{\|s_i - s_j\|^2}{4N_0}\right\}] = E[e^{-X_{ij}/4N_0}]$$

$$\begin{aligned} E[e^{-X_{ij}/4N_0}] &= \sum_{m=0}^{N} \binom{N}{m} 2^{-N} e^{-m4E/4N_0} \\ &= 2^{-N} \sum_{m=0}^{N} \binom{N}{m} (e^{-E/N_0})^m \\ &= 2^{-N}(1+e^{-E/N_0})^N \\ &= \exp_2[-N\left(1-\log_2(1+e^{-E/N_0})\right)] \end{aligned}$$

Let $R_0 = 1 - \log_2\ (1+e^{-E/N_0})$ then

$$\overline{P}_{e,i} \leq \sum_{j\neq i} 2^{-NR_0} = (M-1)2^{-NR_0} \leq M2^{-NR_0} = 2^{-N(R_0-R)}$$

where $R = \frac{\log_2 M}{N}$ is the number of bits transmitted per dimension and E is the signal energy per dimension. We have shown that there exist a signal set for which the average value of the error probability for the $i-th$ signal is small. Thus we have shown that as N goes to ∞ the error probability given s_i was transmitted goes to zero if the rate is less than the cutoff rate R_0. This however does not imply that there exist a code $s_0, ..., s_{M-1}$ such that $P_{e,0}, ..., P_{e,M-1}$ are simultaneously small. It is possible that $P_{e,i}$ is small for some code for which $P_{e,j}$ is large. We now show that we can simultaneously make each of the error probabilities small simultaneously. First chose a code with $M = (2)2^{RN}$ codewords for which the average error probability is less than say $\varepsilon_N/2$ for large N. If more than 2^{NR} of these codewords has $P_{e,i} \geq \varepsilon_N$ then the average error probability would be greater than $\varepsilon_N/2$, a contradiction. Thus at least $M/2 = 2^{NR}$ of the codewords must have $P_{e,i} \leq \varepsilon_N$. So delete the codewords that have $P_{e,i} \geq \varepsilon_N$ (less than half). We obtain a code with (at least) 2^{NR} codewords with $P_{e,i} \to 0$ as $n \to \infty$ for $R < R_0$.

Thus we have proved the following.

<u>Theorem</u>: There exist a signal set with M signals in N dimensions with $P_e \leq 2^{-N(R_0-R)}$ $(\Rightarrow P_e \to 0\ as\ N \to \infty\ provided\ R < R_0)$.

Note: E is the energy per dimension. Each signal then has energy NE and is transmitting $\log_2 M$ bits of information so that $E_b = \frac{NE}{\log_2 M} = E/R$ is the energy per bit of information.

From the theorem, reliable communication ($P_e \to 0$) is possible provided $R < R_0 \leq 1$, i.e.

$$1 - \log_2\ (1+\exp\ \{-E_bR/N_0\}) > R$$

$$1 - R > \log_2(1+e^{-E_bR/N_0})$$

$$2^{1-R} > 1 + e^{-E_bR/N_0} \Rightarrow e^{-E_bR/N_0} < 2^{1-R} - 1$$
$$-E_bR/N_0 < \ln(2^{1-R}-1) \Rightarrow \frac{E_b}{N_0} \geq -\frac{\ln(2^{1-R}-1)}{R}$$

For

$$R \to 0 \qquad -\frac{\ln(2^{1-R}-1)}{R} \to 2\ln 2 \Rightarrow P_e \to 0 \text{ if } E_b/N_0 > 2\ln 2$$

Note: M orthogonal signals have $P_e \to 0$ if $E_b/N_0 > \ln 2$. The rate of orthogonal signals is

$$R = \frac{\log_2 M}{N} = \frac{\log_2 M}{M} \to 0 \text{ as } M \to \infty$$

The theorem guarantees existence of signals with $\frac{\log_2 M}{N} = R > 0$ and $P_e \to 0$ as $M \to \infty$.

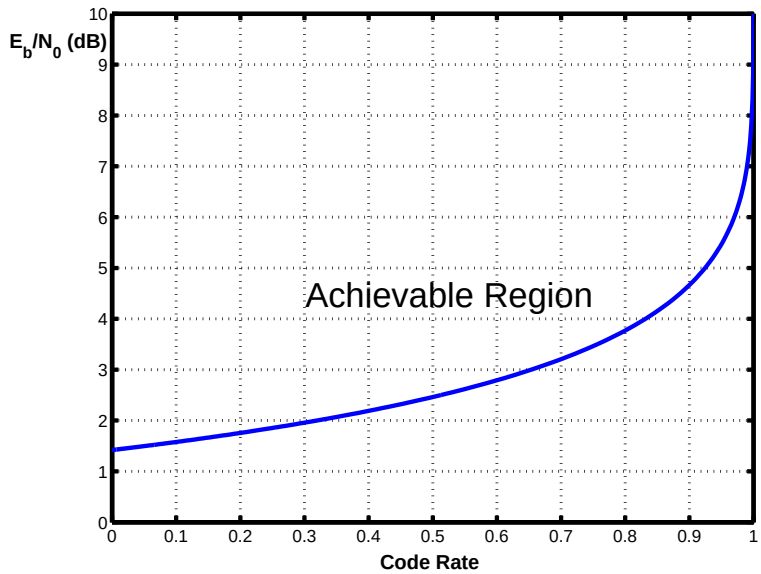


Figure 3.3: Cutoff Rate for Binary Input-Continuous Output Channel

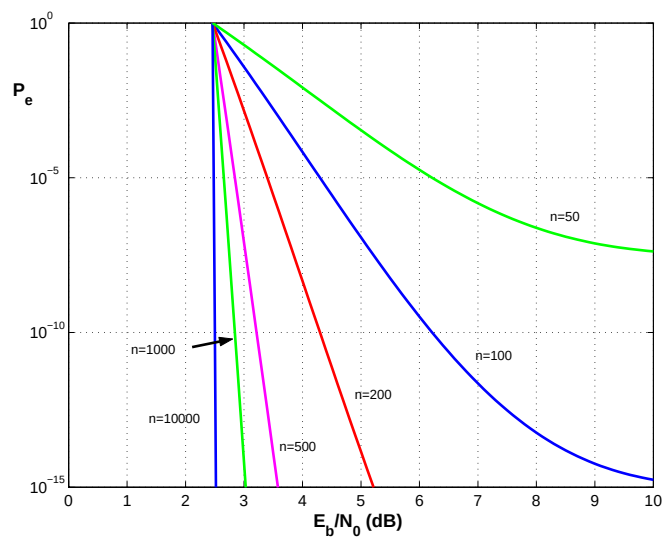


Figure 3.4: Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 1/2 codes

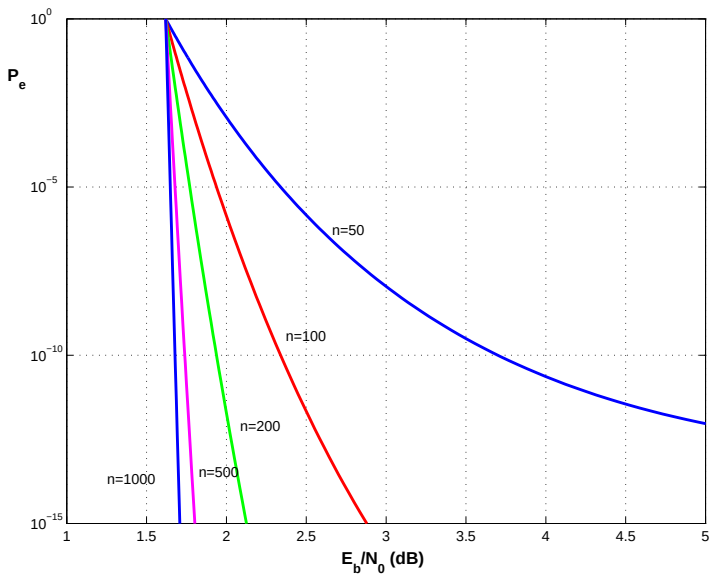


Figure 3.5: Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 1/8 codes

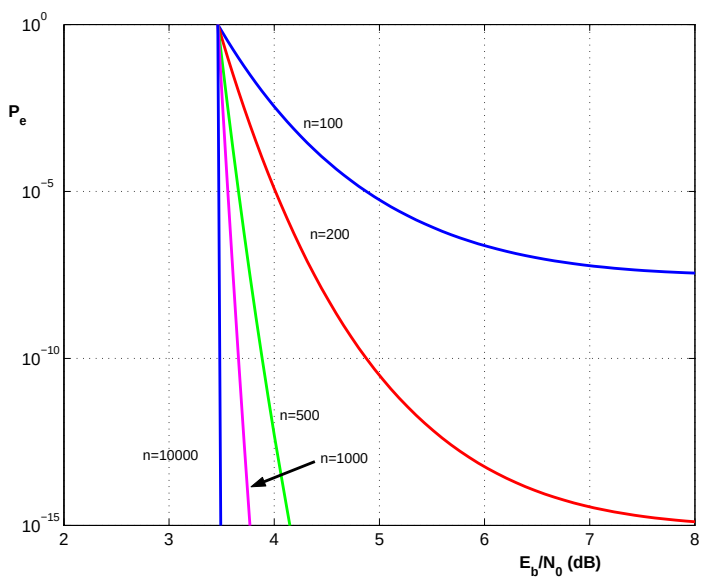


Figure 3.6: Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 3/4 codes

1. Example of Gallager bound for M-ary signals in AWGN.

In this section we evaluate the Gallager bound for an arbitrary signal set in additive white Gaussian noise channel. As usual assume the signal set transmitted has the form

$$s_i(t) = \sum_{j=0}^{N-1} \mu_{i,j}\phi_j(t), \quad i = 0,1,...,M-1$$

The optimal receiver does a correlation with each of the orthonormal functions to produce the decision statistic $r_0,...,r_{N-1}$. The conditional density function of r_i given signal $s_i(t)$ transmitted is given by

$$p_i(\mathbf{r}) = \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_0}} e^{-(r_k-\mu_{i,k})^2/N_0}$$

If we substitute this into the general form of the Gallager bound we obtain

$$\begin{aligned}
P_{e,i} &\leq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}e^{-(r_k-\mu_{i,k})^2/N_0}\right]^{\frac{1}{1+\rho}} \\
&\qquad \left\{\sum_{j\neq i}\left[\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}e^{-(r_k-\mu_{j,k})^2/N_0}\right]^{\frac{1}{1+\rho}}\right\}^{\rho} d\mathbf{r} \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\left(\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}e^{-(r_k^2-2r_k\mu_{i,k}+\mu_{i,k}^2)/N_0(1+\rho)}\right)\right] \\
&\qquad \left\{\sum_{j\neq i}\left[\prod_{k=1}^{N}e^{-(r_k^2-2r_k\mu_{j,k}+\mu_{j,k}^2)/N_0}\right]^{\frac{1}{1+\rho}}\right\}^{\rho} d\mathbf{r} \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\{-\frac{(r_k^2-2r_k\mu_{i,k}+\mu_{i,k}^2)}{N_0(1+\rho)}-\frac{(r_k)^2\rho}{N_0(1+\rho)}\} \\
&\qquad \left\{\sum_{j\neq i}\prod_{k=1}^{N}\exp\{-\frac{(-2r_k\mu_{j,k}+\mu_{j,k}^2)}{N_0(1+\rho)}\}\right\}^{\rho} d\mathbf{r} \\
&= \exp\{-\frac{||\mu_i||^2\rho}{N_0(1+\rho)^2}\}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\{-\frac{1}{N_0}(r_k-\frac{\mu_{i,k}}{1+\rho})^2\} \\
&\qquad \left\{\sum_{j\neq i}\exp\{-\frac{||\mu_j||^2}{N_0(1+\rho)}\}\prod_{k=1}^{N}\exp\{\frac{2r_k\mu_{j,k}}{N_0(1+\rho)}\}\right\}^{\rho} d\mathbf{r} \\
&\leq \exp\{-\frac{||\mu_i||^2\rho}{N_0(1+\rho)^2}\}\left\{\sum_{j\neq i}\exp\{-\frac{||\mu_j||^2}{N_0(1+\rho)}\}\right. \\
&\qquad \left.\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\{-\frac{1}{N_0}(r_k-\frac{\mu_{i,k}}{1+\rho})^2\}\right)\exp\{\frac{2r_k\mu_{j,k}}{N_0(1+\rho)}\}d\mathbf{r}\right\}^{\rho} \\
&= \exp\{-\frac{||\mu_i||^2\rho}{N_0(1+\rho)^2}\}\left\{\sum_{j\neq i}\exp\{-\frac{||\mu_j||^2}{N_0(1+\rho)}\}\exp\{-\frac{1}{N_0}\left[\frac{||\mu_i||^2}{(1+\rho)^2}-\frac{||\mu_i+\mu_j||^2}{(1+\rho)^2}\right]\}\right\}^{\rho} \\
&= \left(\sum_{j\neq i}\exp\left\{-\frac{1}{N_0}\left[\frac{d_E^2(\mu_i,\mu_j)}{(1+\rho)^2}-\frac{(1-\rho)||\mu_j||^2}{(1+\rho)^2}\right]\right\}\right)^{\rho}
\end{aligned}$$

When the signals are all orthogonal to each other then $d_E^2 = 2E$ for $i \neq j$ and $||\mu_j||^2 = E$ and the bound becomes

$$P_{e,i} \leq \left(\sum_{j\neq i}\exp\left\{-\frac{1}{N_0}\left[\frac{2E}{(1+\rho)^2}-\frac{(1-\rho)E}{(1+\rho)^2}\right]\right\}\right)^{\rho}$$

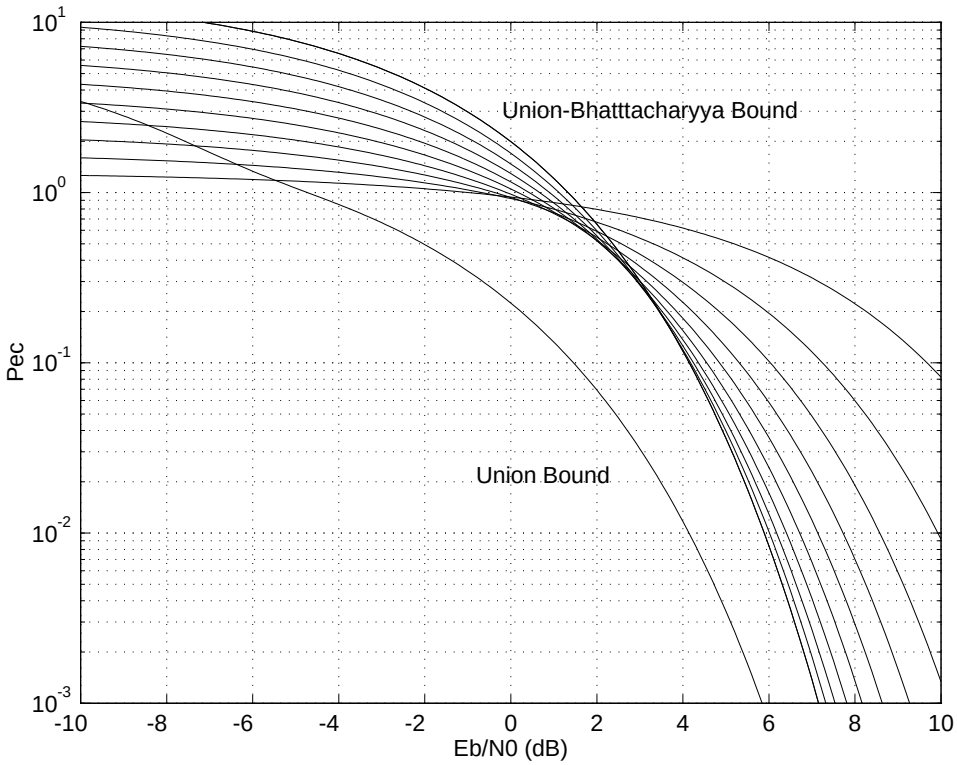


Figure 3.7: Comparison of Gallager Bound, Union Bound and Union Bhattacharyya Bound for the Hamming Code with BPSK Modulation

$$= \quad (M-1)^{\rho} \exp\left\{-\frac{E\rho}{N_0(1+\rho)}\right\}$$

This is identical to the previous expression.

Now we consider a couple of different signal sets. The first signal set has 16 signals in seven dimensions. The energy in each dimension is E so the total energy transmitted is $7E$. The energy transmitted per information bit is $E_b = 7E/4$ The geometry of the signal set is such that for any signal there are seven other signals at Euclidean distance $12E$, seven other signals at Euclidean distance $16E$ and one other signal at distance $28E$. All signals have energy $7E$. (This is called the Hamming code). The fact that the signal set is geometrically uniform is due to the linearity of the code. We plot the Gallager bound for $\rho = 0.1, 0.2, ..., 1.0$. The Union-Bhattacharyya bound is the Gallger bound with $\rho = 1.0$. The second signal set has 256 signals in 16 dimensions with 112 signals at distance $24E$, 30 signals at distance $32E$, 112 signals at distance $40E$ and 1 signal at distance $64E$. In this case $E_b = 2E$.

As can be seen from the figures the union bound is the tightest bound except at very low signal-to-noise ratios where the Gallager bound stays below 1. At reasonable signal-to-noise ratios the optimum ρ in the Gallager bound is 1 and thus it reduces to the Union-Bhattacharyya bound.

6. Problems

1. Using L'Hospitals rule on the log of Φ^{M-1} show that if $E = E_b \log_2 M$ then

$$\lim_{M\to\infty} \quad \Phi\left(x + \sqrt{\frac{2E}{N_0}}\right)^{M-1} = \begin{cases} 0, & E_b/N_0 < \ln 2 \\ 1, & E_b/N_0 > \ln 2 \end{cases} \tag{3.1}$$

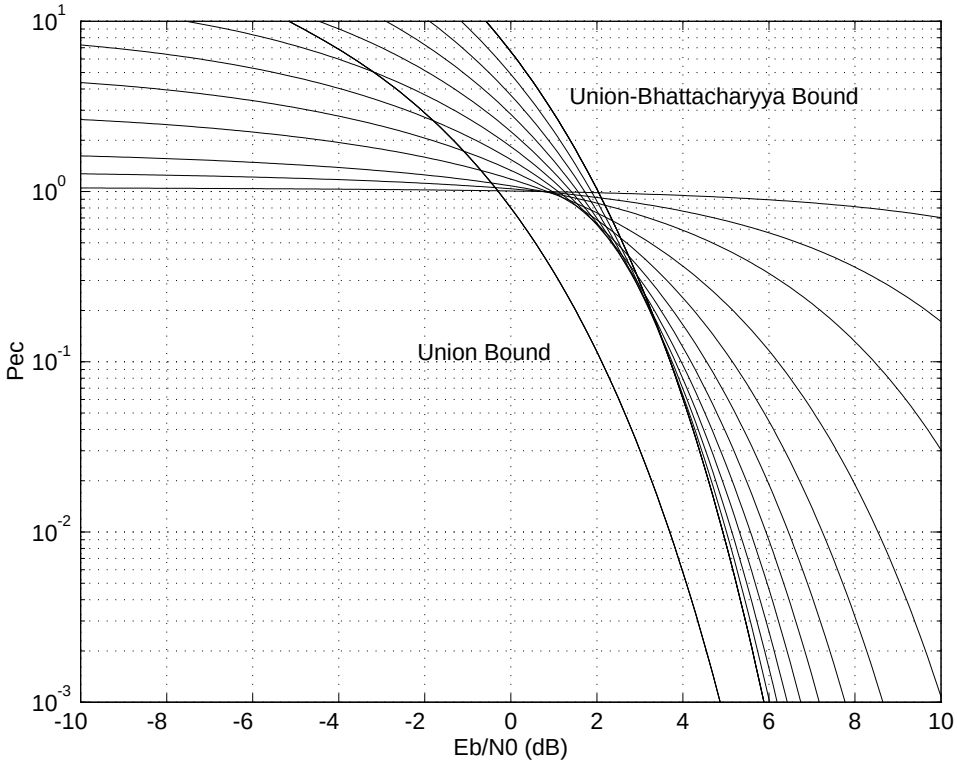


Figure 3.8: Comparison of Gallager Bound, Union Bound and Union Bhattacharyya Bound for the Nordstrom-Robinson code with BPSK Modulation

and consequently $\lim_{M\to\infty} P_e$ is 0 if $E_b/N_0 > \ln 2$ and 1 if $E_b/N_0 < \ln 2$ where P_e is the error probability for M orthogonal signals.

2. (a) Show that for any equienergy, equidistant (real) signal set, $(s_i, s_j) =$ a constant for $i \neq j$. (Note: equienergy implies $\| s_i \|^2$ is a constant and equidistant implies $\| s_i - s_j \|$ is a constant).

   (b) For any equienergy signal set show that

$$\rho_{ave} = \frac{1}{M(M-1)} \sum_{i=0}^{M-1} \sum_{\substack{j=0 \\ j\neq i}}^{M-1} \rho_{ij} \geq -\frac{1}{M-1}$$

   where

$$\rho_{ij} = (s_i, s_j)/(||s_i||||s_j||) = (s_i, s_j)/E$$

3. (R_0 coding theorem for discrete memoryless channels)

   Consider the discrete memoryless channel (DMC) which has input alphabet $A = \{a_1, a_2, \ldots, a_{|A|}\}$ (with $|A|$ being the number of letters in A and is finite) and output alphabet $B = \{b_1, b_2, \ldots, b_{|B|}\}$ (with $|B|$ being the number of letters in B and is finite). As in the Gaussian noise channel the channel is characterized by a set of "transition probabilities" $p(b|a)$, $a \in A$, $b \in B$ such that if we transmit a signal $s_i(t)$ where

$$s_i(t) = \sum_{l=1}^{N} s_{i,l}\phi_l(t)$$

   with $s_{i,l} \in A$ then the received signal has the form

$$r(t) = \sum_{l=1}^{N} r_l \phi_l(t)$$

   with $p\{r_l = b | s_{i,l} = a\} = p(b|a)$ for $a \in A$, $b \in B$ and

$$p\{r_1, \ldots, r_N | s_{i,1}, \ldots, s_{i,N}\} = \prod_{l=1}^{N} p(r_l | s_{i,l}) \qquad (1).$$

   Now we come to the R_0 coding theorem for a discrete (finite alphabet) memoryless (equation (1) is satisfied) channel (DMC).

   Prove there exist M signals in N dimensions such that

$$P_e = \frac{1}{M} \sum_{i=0}^{M-1} P_{e,i} \leq 2^{-N(R_0 - R)}$$

   where

$$R = \frac{\log_2 M}{N},$$
$$R_0 = -\log_2 J_0,$$
$$J_0 = \min_{p(x)} E[J(X_1, X_2)] \qquad (2)$$
$$J(a_1, a_2) = \sum_{b\in B} \sqrt{p(b|a_1)p(b|a_2)}$$

   and in (2) X_1 and X_2 are independent, identically distributed random variables with common distribution $p(x)$,

   Step 1: Let $M = 2$ and let

$$\mathbf{s}_0 = (s_{0,1}, \ldots, s_{0,N})$$

and

$$\mathbf{s}_1 = (s_{1,1}, \ldots, s_{1,N})$$

The decoder will not decide s_0 if

$$p(\mathbf{r}|\mathbf{s}_1) > p(\mathbf{r}|\mathbf{s}_0)$$

Let

$$R_1 = \{\mathbf{r} : p(\mathbf{r}|\mathbf{s}_1) \geq p(\mathbf{r}|\mathbf{s}_0)\}$$

and

$$R_0 = \{\mathbf{r} : p(\mathbf{r}|\mathbf{s}_0) \geq p(\mathbf{r}|\mathbf{s}_1)\}$$

(Note that R_0 and R_1 may not be disjoint). Show that

$$\begin{aligned} P_{e,i} &= \sum_{\mathbf{r} \in R_j} p(\mathbf{r}|\mathbf{s}_i) \quad i = 0,1,\ j \neq i \\ &\leq \sum_{all\ \mathbf{r}} \sqrt{p(\mathbf{r}|\mathbf{s}_0) p(\mathbf{r}|\mathbf{s}_1)} \\ &= \prod_{l=1}^{N} \sum_{r_l \in B} \sqrt{p(r_l|s_{0l}) p(r_l|s_{1l})} \\ &= \prod_{l=1}^{N} J(s_{0,l}, s_{1,l}) \end{aligned}$$

Step 2: Apply the union bound to obtain for M signals (codewords)

$$P_{e,i} \leq \sum_{j \neq i} \prod_{l=1}^{N} J(s_{i,l}, s_{j,l})$$

Step 3: Average $P_{e,i}$ over all possible signal sets where the signals are chosen independently according to the distribution that achieves J_0 (i.e. that distribution $p(x)$ on A such that

$$J_0 = E\left[J(X_1, X_2)\right].$$

(treat $s_{i,j} : 1 \leq i \leq M, 1 \leq j \leq N$ as i.i.d. random variables with distribution $p(x)$) to show that

$$\bar{P}_{e,i} \triangleq E\left[P_{e,i}\right] \leq M 2^{-NR_0}.$$

Step 4: Complete the proof.

4. Show for a binary symmetric channel defined as

$$A = B = \{0,1\} \quad p(b|a) = \begin{cases} p & a \neq b \\ 1-p & a = b \end{cases}$$

that

$$R_0 = 1 - \log_2\left[1 + 2\sqrt{p(1-p)}\right].$$

5. A set of 16 signals is constructed in 7 dimension using only two possible coefficients, i.e. $s_{i,j} \in \left\{+\sqrt{E}, -\sqrt{E}\right\}$. Let $A_k = \left|\left\{(i,j) : |s_i - s_j|^2 = 4Ek\right\}\right|$ i.e. A_k is the number of signal pairs with squared distance $4Ek$. The signals are chosen so that

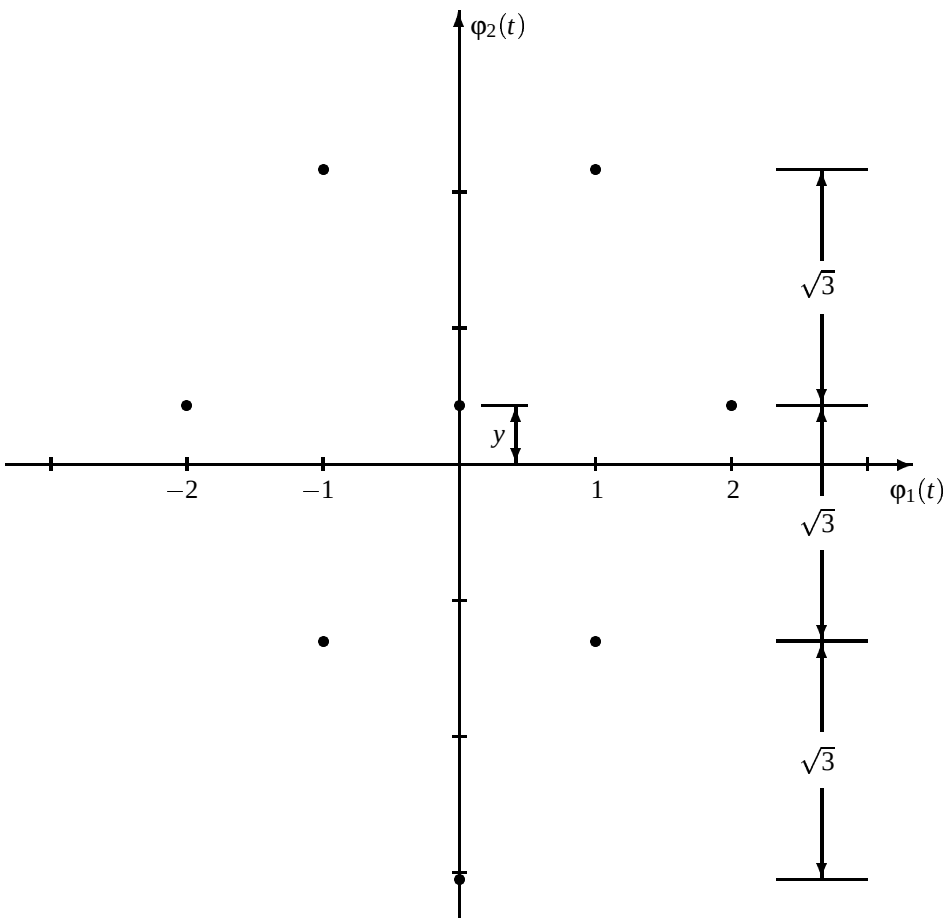
$$A_k = \begin{cases} 16 & k = 0 \\ 0 & k = 1,2 \\ 112 & k = 3,4 \\ 0 & k = 5,6 \\ 16 & k = 7 \end{cases}$$

Find the union bound and the union-Bhattacharyya bound on the error probability of the optimum receiver in additive white Gaussian noise with two sided power spectral density $N_0/2$.

6. A modulator uses two orthonormal signals ($\phi_1(t)$ and $\phi_2(t)$) to transmit 3 bits of information (8 possible equally likely signals) over an additive white Gaussian noise channel with power spectral density $N_0/2$. The signals are given as

$$\begin{aligned}
s_1(t) &= \sqrt{E}(-1\phi_1(t) + (y+\sqrt{3})\phi_2(t)) \\
s_2(t) &= \sqrt{E}(1\phi_1(t) + (y+\sqrt{3})\phi_2(t)) \\
s_3(t) &= \sqrt{E}(-2\phi_1(t) + (y)\phi_2(t)) \\
s_4(t) &= \sqrt{E}(0\phi_1(t) + y\phi_2(t)) \\
s_5(t) &= \sqrt{E}(2\phi_1(t) + (y)\phi_2(t)) \\
s_6(t) &= \sqrt{E}(-1\phi_1(t) + (y-\sqrt{3})\phi_2(t)) \\
s_7(t) &= \sqrt{E}(1\phi_1(t) + (y-\sqrt{3})\phi_2(t)) \\
s_8(t) &= \sqrt{E}(0\phi_1(t) + (y-2\sqrt{3})\phi_2(t))
\end{aligned}$$

(a) Determine the optimum value of the parameter y to minimize the average signal power transmitted. Draw the optimum decision regions for the signal set (in two dimensions). (b) Determine the union bound to the average error probability in terms of energy per information bit E_b to noise density ratio N_0. (c) Can you tighten this bound?

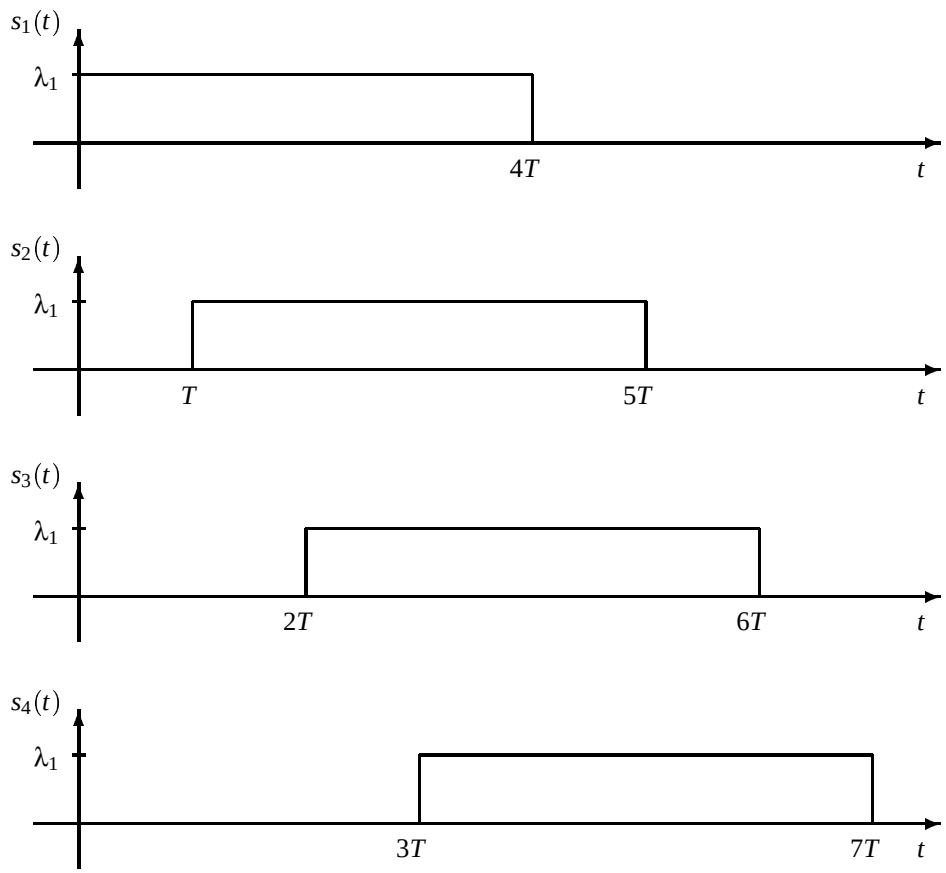


7. Consider an additive (nonwhite) Gaussian noise channel with one of two signals transmitted. Assume the noise has covariance function $K(s,t)$. Using the Bhattacharray bound show that the error probability when transmitting one of two signals ($s_0(t)$ or $s_1(t)$) can be bounded by

$$P_e \leq e^{-||K^{-1/2}(s_0-s_1)||^2/8}.$$

If the noise is white, what does the bound become?

8. Consider a Poisson channel with one of 4 signals transmitted. Let the signals be as shown below. Assume when the signal is present that the intensity of the photon process is λ_1 and when the signal is not present the intensity is λ_0. That is the received signal during the interval $[0,T]$ is Poisson with parameter λ_1 if the laser is on and λ_0 if the laser is off. Find the optimal receiver for minimizing the probability of error for a signal (as opposed to a bit). Find an upper bound on the error probability.



9. A signal set consists of 256 signals in 16 dimensions with the coefficients being either $+\sqrt{E}$ or $-\sqrt{E}$. The distance structure is given as

$$A_k = |\{(i,j) : ||s_i - s_j||^2 = 4Ek\}| = \begin{cases} 256 & k=0 \\ 28672 & k=6 \\ 7680 & k=8 \\ 28672 & k=10 \\ 256 & k=16 \\ 0 & \text{otherwise} \end{cases}$$

These signals are transmitted with equal probability over an additive white Gaussian noise channel. De-

termine the union bound on the error probability. Determine the union-Bhattacharyya bound on the error probability. Express your answer in terms of the energy transmitted per bit. What is the rate of the code in bits/dimension?

10. A communication system uses a N dimensions and a code rate of R bits/dimension. The goal is not low error probability but high throughput (expected number of successfully received information bits per coded bit in a block on length N). If we use a low code rate then we have high success probability for a packet but few information bits. If we use a high code rate then we have a low success probability but a larger number of bits transmitted. Assume the channel is an additive white Gaussian noise channel and the input is restricted to binary modulation (each coefficient in the orthonormal expansion is either $+\sqrt{E}$ or $-\sqrt{E}$. Assume as well that the error probability is related to the block length, energy per dimension and code rate via the cutoff rate theorem (soft decisions). Find (and plot) the throughput for code rates varying from 0.1 to 0.9 in steps of 0.1 as a function of the energy per information bit. (Use Matlab to plot the throughput). Assume $N = 500$. Be sure to normalize the energy per coded bit to the energy per information bit. Compare the throughput of hard decision decoding (BPSK and AWGN) and soft decision decoding.