Chapter 4

Asymptotic Performance

In this chapter we discuss the asymptotic performance of signals. First we consider the case of M signals in N dimension when transmitted over the additive white Gaussian noise channel. We let M and N become large and show that the error probability goes to zero provide the rate $\log_2(M)/N$ in bits per dimension is smaller than a quantity that depends on the signal energy available and the noise power density. In addition we will show that if the rate is greater than this same number the error probability is bounded away from zero. Next we consider the case where the coefficients in the orthonormal expansion of the signal set are taken from a finite alphabet. Furthermore the output is quantized. Finally we summarize the results for various channels.

1. Capacity Theorem

There exist a constant $C = \frac{1}{2}\log\left(1+\frac{2E}{N_0}\right)$ such that

(a) If $\frac{\log_2 M}{N} > C$ then *every* set of M signals in N dimensions (of energy $\leq EN$) when used on a AWGN channel has error probability close to 1 for M, N large.

(b) If $\frac{\log_2 M}{N} < C$ then there exist a set of M signal in N dimensions (of energy EN) such that when used on the AWGN channel the error probability is close to 0 for M, N large.

Let $R = \frac{\log_2 M}{N} \Rightarrow M = 2^{RN}$ consider M signals with $\sum_{l=1}^{N} s_{i,l}^2 \leq NE$.

Let

$$\begin{aligned}
\tilde{s}_{i,l} &= s_{i,l}/\sqrt{N} \;\Rightarrow\; \|\tilde{\mathbf{s}}_i\|^2 = \|\mathbf{s}_i\|^2/N \leq E \\
\tilde{n}_l &= n_l/\sqrt{N} \;\Rightarrow\; E[\tilde{n}_l^2] = N_0/2N \\
\tilde{\mathbf{r}} &= \tilde{\mathbf{s}}_i + \tilde{\mathbf{n}} \;.
\end{aligned}$$

Note that $\|\tilde{\mathbf{n}}\|^2$ is a random variable with mean

$$\begin{aligned}
E\left[\|\tilde{\mathbf{n}}\|^2\right] &= E\left[\sum_{l=1}^{N} \tilde{n}_l^2\right] = E\left[\sum_{l=1}^{N} n_l^2/N\right] \\
&= \frac{1}{N}\sum_{l=1}^{N} E[n_l^2] = \frac{N_0}{2} \quad \text{(independent of } N\text{)}
\end{aligned}$$

and variance

$$\mathrm{Var}\left[\|\tilde{\mathbf{n}}\|^2\right] = \mathrm{Var}\left[\sum_{l=1}^{N} \tilde{n}_l^2\right] = \mathrm{Var}\left(\sum_{l=1}^{N} n_l^2/N\right)$$

$$
\begin{aligned}
&= \sum_{l=1}^{N} \mathrm{Var}(n_l^2/N) = \frac{1}{N^2}\sum_{l=1}^{N}\mathrm{Var}(n_l^2) \\
\mathrm{Var}[n_l^2] &= E[n_l^4] - E^2[n_l^2] = E[n_l^4] - \sigma^4 \\
E[n_l^4] &= 3\sigma^4 \qquad \text{(eq. 1.1.97 of Proakis)} \\
\mathrm{Var}[n_l^2] &= 2\sigma^4 \\
\mathrm{Var}\left[\|\tilde{\mathbf{n}}\|^2\right] &= \frac{2\sigma^4}{N} = \frac{2}{N}\left(\frac{N_0}{2}\right)^2 \to 0 \text{ as } N \to \infty
\end{aligned}
$$

Chebyshev inequality (Proakis, eq. 1.1.153)

$$
P\left\{\left|\|\tilde{\mathbf{n}}\|^2 - \frac{N_0}{2}\right| \geq \Delta\right\} \leq \frac{2}{N}\left(\frac{N_0}{2}\right)^2 \frac{1}{\Delta^2} \to 0 \text{ as } N \to \infty
$$

Thus the noise vector has length $\frac{N_0}{2}$ for large N. This is called sphere hardening.

Intuition: For N large the noise vector $\tilde{\mathbf{n}}$ will have length $\frac{N_0}{2}$ so the decision region must include the sphere of radius $\frac{N_0}{2}$. If the number of signals is too large there will be overlapping of spheres.

Proof of (a): Preliminaries:

$$
\begin{aligned}
\|\tilde{\mathbf{r}}\| &= \|\tilde{\mathbf{s}}_i + \tilde{\mathbf{n}}\| \\
\|\tilde{\mathbf{r}}\|^2 &= \|\tilde{\mathbf{s}}_i + \tilde{\mathbf{n}}\|^2 \\
&= \|\tilde{\mathbf{s}}_i\|^2 + 2(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}) + \|\tilde{\mathbf{n}}\|^2 \\
&= \|\tilde{\mathbf{s}}_i\|^2 + \frac{2}{N}\sum_{l=1}^{N} s_{i,l} n_l + \|\tilde{\mathbf{n}}\|^2 \quad .
\end{aligned}
$$

$$
\begin{aligned}
E\left[\|\tilde{\mathbf{r}}\|^2\right] &= \|\tilde{\mathbf{s}}_i\|^2 + E\left[\|\tilde{\mathbf{n}}\|^2\right] \\
&= \|\tilde{\mathbf{s}}_i\|^2 + \frac{N_0}{2} \\
&\leq E + N_0/2 \quad .
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left[\|\tilde{\mathbf{r}}\|^2\right] &= \mathrm{Var}\left[\|\tilde{\mathbf{s}}_i + \tilde{\mathbf{n}}\|^2\right] \\
&= \mathrm{Var}\left[\sum_{l=1}^{N}(\tilde{s}_{i,l} + \tilde{n}_l)^2\right] \\
&= \sum_{l=1}^{N}\mathrm{Var}[(\tilde{s}_{i,l} + \tilde{n}_l)^2] \\
&= \sum_{l=1}^{N} E\left[\left((\tilde{s}_{i,l} + \tilde{n}_l)^2 - E(\tilde{s}_{i,l} + \tilde{n}_l)^2\right)^2\right] \\
&= \sum_{l=1}^{N} E\left(\left[\tilde{s}_{i,l}^2 + 2\tilde{s}_{i,l}\tilde{n}_l + \tilde{n}_l^2 - E[\tilde{s}_{i,l}^2 + 2\tilde{s}_{i,l}\tilde{n}_l + \tilde{n}_l^2]\right]^2\right) \\
&= \sum_{l=1}^{N} E\left[\left(2\tilde{s}_{i,l}\tilde{n}_l + \tilde{n}_l^2 - E[\tilde{n}_l^2]\right)^2\right] \qquad \downarrow (a+b)^2 \leq 2a^2 + 2b^2 \\
&\leq \sum_{l=1}^{N} 2\left[E[(2\tilde{s}_{i,l}\tilde{n}_l)^2] + E\left[(\tilde{n}_l^2 - E[\tilde{n}_l^2])^2\right]\right] \\
&= 2\left[\sum_{l=1}^{N} 4\tilde{s}_{i,l}^2 E[\tilde{n}_l^2] + \mathrm{Var}[\tilde{n}_l^2]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= (8\sum_{l=1}^{N} \tilde{s}_{i,l}^2 \frac{N_0}{2N}) + 2\ \mathrm{Var}\ [\|\tilde{\mathbf{n}}\|^2] \\
&\leq \frac{4EN_0}{N} + \frac{4}{N}\left(\frac{N_0}{2}\right)^2 = \frac{1}{N}\left[4EN_0 + 4\left(\frac{N_0}{2}\right)^2\right] \to 0 \text{ as } N \to \infty.
\end{aligned}
$$

Thus $\|\tilde{r}\|^2$ has mean $\leq E + \frac{N_0}{2}$ and variance $\leq \frac{1}{N}[4EN_0 + N_0^2] \to 0$ as $N \to \infty$. Thus

$$
\begin{aligned}
P\left\{\left|\|\tilde{\mathbf{r}}\|^2 - E[\|\tilde{\mathbf{r}}\|^2]\right| \geq \Delta\right\} &\leq \frac{\mathrm{Var}(\|\tilde{\mathbf{r}}\|^2)}{\Delta^2} \\
&\leq \frac{4EN_0 + N_0^2}{N\Delta^2}
\end{aligned}
$$

$$
P\left\{\|\tilde{\mathbf{r}}\|^2 \geq E\left[\|\tilde{\mathbf{r}}\|^2\right] + \Delta\right\} \leq \frac{4EN_0 + N_0^2}{N\Delta^2}
$$

$$
P\left\{\|\tilde{\mathbf{r}}\|^2 \geq E + \frac{N_0}{2} + \Delta\right\} \leq P\left\{\|\tilde{\mathbf{r}}\|^2 \geq E\left[\|\tilde{\mathbf{r}}\|^2\right] + \Delta\right\} \leq \frac{4EN_0 + N_0^2}{N\Delta^2}
$$

Let I_r be a sphere of radius $\sqrt{E + \frac{N_0}{2} + \Delta}$ centered at origin

$$
P\{\text{correct}\} = P\{\text{correct}, \tilde{\mathbf{r}} \in I_r\} + P\{\text{correct}, \tilde{\mathbf{r}} \notin I_r\}
$$

$$
P\{\text{correct}, \tilde{\mathbf{r}} \notin I_r\} \leq P\{\tilde{\mathbf{r}} \notin I_r\} \leq \frac{4EN_0 + N_0^2}{N\Delta^2} \to 0 \text{ as } N \to \infty
$$

Let $\hat{R}_i = R_i \cap I_r$

$$
P\{\text{correct}, r \in I_r\} = \sum_{i=0}^{M-1} P\{\hat{R}_i | H_i\} P\{H_i\} = \frac{1}{M}\sum_{i=0}^{M-1} P\{\hat{R}_i | H_i\}
$$

Notation and Facts

$$
\begin{aligned}
I_r &\triangleq \text{sphere of radius } r \text{ centered at origin} \\
V_r &\triangleq \text{volume of sphere of radius } r \\
&= B_N r^N.
\end{aligned}
$$

For $N = 2$, and $N = 3$ the coefficients are $B_3 = 4/3\,\pi$ and $B_2 = \pi$. In general

$$
B_N = \begin{cases} \frac{2^N \pi^{\frac{N-1}{2}} \left(\frac{N-1}{2}\right)!}{N!} & N \text{ odd} \\ \frac{\pi^{N/2}}{\left(\frac{N}{2}\right)!} & N \text{ even.} \end{cases}
$$

Steps in proof of (a): Let $r = \sqrt{E + \frac{N_0}{2} + \Delta}$

1) $P\{\text{ correct }\} = P\{\text{ correct }, \tilde{\mathbf{r}} \in I_r\} + P\{\text{ correct}, \tilde{\mathbf{r}} \notin I_r\}$.

2) $P\{\text{ correct}, \tilde{\mathbf{r}} \notin I_r\} \leq P\{\tilde{\mathbf{r}} \notin I_r\} \leq \dfrac{4EN_0 + N_0^2}{N\Delta^2} \to 0$ as $N \to \infty$.

   (This step uses Chebyshev's Inequality).

3) $P\{\text{correct}, \tilde{\mathbf{r}} \in I_r\} \leq P\left\{\|\tilde{\mathbf{n}}\| \leq \left(B_N^{-1}\frac{V_r}{M}\right)^{1/N}\right\}$.

4) $P\left\{\|\tilde{\mathbf{n}}\| \leq \left(B_n^{-1}\frac{V_r}{M}\right)^{1/N}\right\} \leq \frac{2}{N}\left(\frac{N_0}{2}\right)^2 \frac{1}{\widetilde{\Delta}^2}$ where $\widetilde{\Delta}^2 = \frac{N_0}{2} - \left(\frac{1}{M}\right)^{2/N}\left(E + \frac{N_0}{2} + \Delta\right)$.

5) $\widetilde{\Delta}^2 > 0$ if $R > \frac{1}{2}\log_2\left(1 + \frac{2E}{N_0} + \frac{\Delta}{N_0/2}\right) \Rightarrow N\widetilde{\Delta}^2 \to \infty$ as $N \to \infty$.

6) $P\{\text{correct}\} \to 0$ if $R > \frac{1}{2}\log_2\left(1 + \frac{2E}{N_0}\right)$.

Proof of 1: Obvious!
Proof of 2: $P\{\tilde{\mathbf{r}} \notin I_r\} = P\{\|\tilde{\mathbf{r}}\|^2 > E + \frac{N_0}{2} + \Delta\}$.
Simple Calculations show that

$$E\left[\|\tilde{\mathbf{r}}\|^2\right] \leq E + \frac{N_0}{2}$$

and

$$\mathrm{Var}\left(\|\tilde{\mathbf{r}}\|^2\right) \leq \frac{4EN_0 + N_0^2}{N} \quad .$$

Thus by Chebyshev's inequality we obtain

$$\begin{aligned} P\left\{\|\tilde{\mathbf{r}}\|^2 > E + \frac{N_0}{2} + \Delta\right\} &\leq P\left\{\|\tilde{\mathbf{r}}\|^2 > E\left[\|\tilde{\mathbf{r}}\|^2\right] + \Delta\right\} \\ = P\left\{\|\tilde{\mathbf{r}}\|^2 - E\left[\|\tilde{\mathbf{r}}\|^2\right] > \Delta\right\} &\leq P\left\{\left|\|\tilde{\mathbf{r}}\|^2 - E\left[\|\tilde{\mathbf{r}}\|^2\right]\right| > \Delta\right\} \\ \leq \frac{\mathrm{Var}\left(\|\tilde{\mathbf{r}}\|^2\right)}{\Delta^2} &\leq \frac{4EN_0 + N_0^2}{N\Delta^2} \quad . \end{aligned}$$

Proof of 3: The idea behind the first part of the proof of this step is illustrated in the following figure. Let $\hat{R}_i = R_i \cap I_r$ (R_i is region where H_i is decided). Then the probability of being correct and in I_r given H_i is the probability of being in $\hat{R}_i$. Now consider a sphere with volume equal to the volume of $\hat{R}_i$ and centered at the signal point $\tilde{s}_i$. The probability of the received signal being in the region $\hat{R}_i$ given H_i is less than the probability of being in the sphere having the same volume as $\hat{R}_i$ (the shaded region in the picture). This is because the probability density is decreasing the farther from the signal point. Thus if all the volume in $\hat{R}_i$ but outside the sphere was moved inside the sphere but outside $\hat{R}_i$ the probability would be larger. Thus we can upper bound the probability of correct given H_i.

This is shown mathematically below.

$P\{\text{correct}, \tilde{\mathbf{r}} \in I_r\} = \sum_{i=0}^{M-1} P\{\hat{R}_i | H_i\} P\{H_i\}$. Let $V(\hat{R}_i)$ be volume of region $\hat{R}_i$ and let ρ_i be radius of a sphere of volume $V(\hat{R}_i)$. Let $Q_i = \{\tilde{\mathbf{n}} : \tilde{\mathbf{n}} + \tilde{\mathbf{s}}_i \in \widehat{R}_i\}$. Then

$$\int I(\tilde{\mathbf{r}} \in \hat{R}_i) d\tilde{\mathbf{r}} = \int I(\tilde{\mathbf{n}} \in Q_i) d\tilde{\mathbf{n}} = \int I(\|\tilde{\mathbf{n}}\| \leq \rho_i) d\tilde{\mathbf{n}}.$$

Now

$$\int I(\tilde{\mathbf{n}} \in Q_i) d\tilde{\mathbf{n}} = \int I(\tilde{\mathbf{n}} \in Q_i) I\left(\|\tilde{\mathbf{n}}\| \leq \rho_i\right) d\tilde{\mathbf{n}} + \int I(\tilde{\mathbf{n}} \in Q_i) I\left(\|\tilde{\mathbf{n}}\| > \rho_i\right) d\tilde{\mathbf{n}}.$$

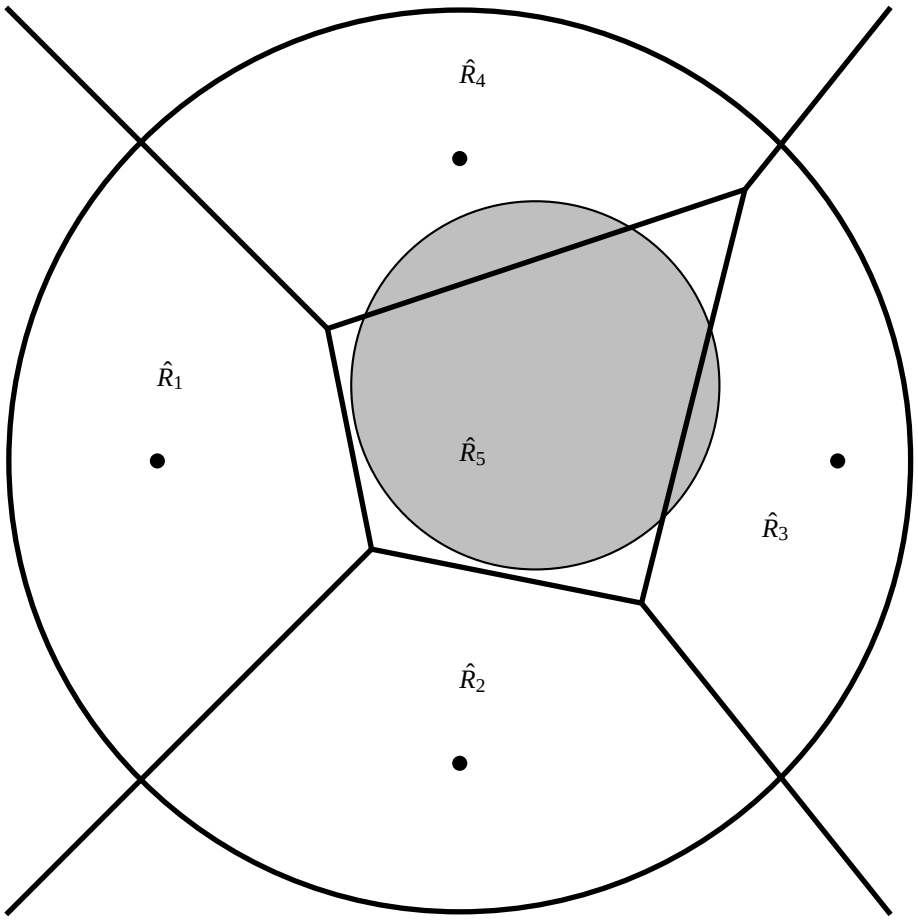


Figure 4.1: Decision Regions of Demodulator

Also

$$\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) d\tilde{\mathbf{n}} = \int I(\tilde{\mathbf{n}} \in Q_i) I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) d\tilde{\mathbf{n}} + \int I(\tilde{\mathbf{n}} \notin Q_i) I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) d\tilde{\mathbf{n}}.$$

Because the left hand side of the above two equations are equal, we have that the right sides are also equal. Thus, after cancelling common terms we obtain

$$\int I(\tilde{\mathbf{n}} \in Q_i) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho_i\Big) d\tilde{\mathbf{n}} = \int I(\tilde{\mathbf{n}} \notin Q_i) I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) d\tilde{\mathbf{n}} \ .$$

If $\|\tilde{\mathbf{n}}\| \le \rho_i$ then $p(\tilde{\mathbf{n}}) \ge p\left(\frac{\rho_i}{\sqrt{N}} \cdot \underline{1}\right)$ where

$$\frac{\rho_i}{\sqrt{N}} \cdot 1 = \left(\frac{\rho_i}{\sqrt{N}}, \frac{\rho_i}{\sqrt{N}}, \ldots, \frac{\rho_i}{\sqrt{N}}\right)$$

If

$$\|\tilde{\mathbf{n}}\| \ge \rho_i \ \text{ then } \ p(\tilde{\mathbf{n}}) \le p\left(\frac{\rho_i}{\sqrt{N}} \cdot \underline{1}\right)$$

So

$$\begin{aligned} \int I(\tilde{\mathbf{n}} \in Q_i) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho_i\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} &\le \int I(\tilde{\mathbf{n}} \in Q_i) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho_i\Big) p\left(\frac{\rho_i}{\sqrt{N}} \cdot \underline{1}\right) d\tilde{\mathbf{n}} \\ &= \int I(\tilde{\mathbf{n}} \notin Q_i) I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) p\left(\frac{\rho_i}{\sqrt{N}} \cdot \underline{1}\right) d\tilde{\mathbf{n}} \end{aligned}$$

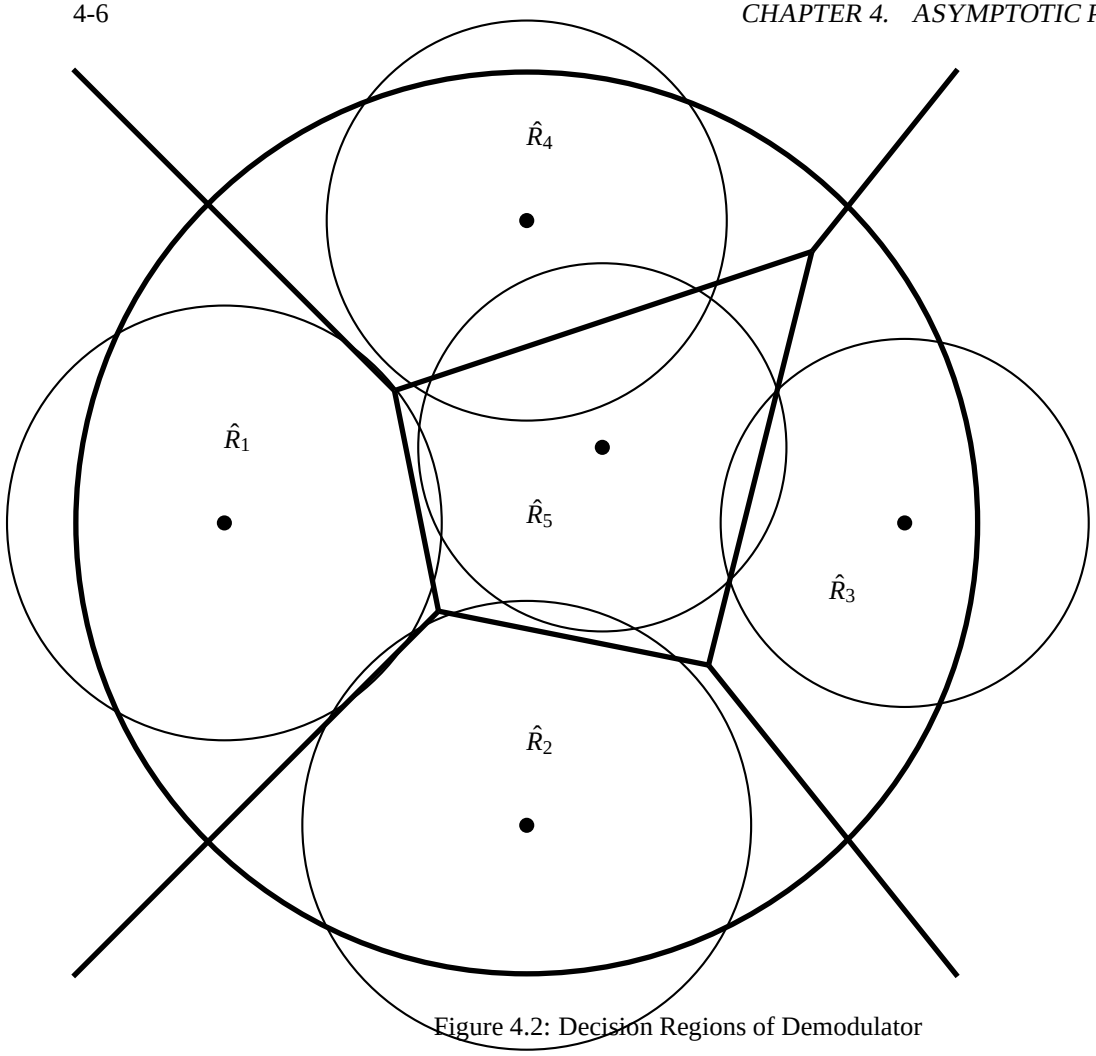


Figure 4.2: Decision Regions of Demodulator

$$\leq \int I(\tilde{\mathbf{n}} \notin Q_i) I\Big(\|\tilde{\mathbf{n}}\| \leq \rho_i\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}}$$

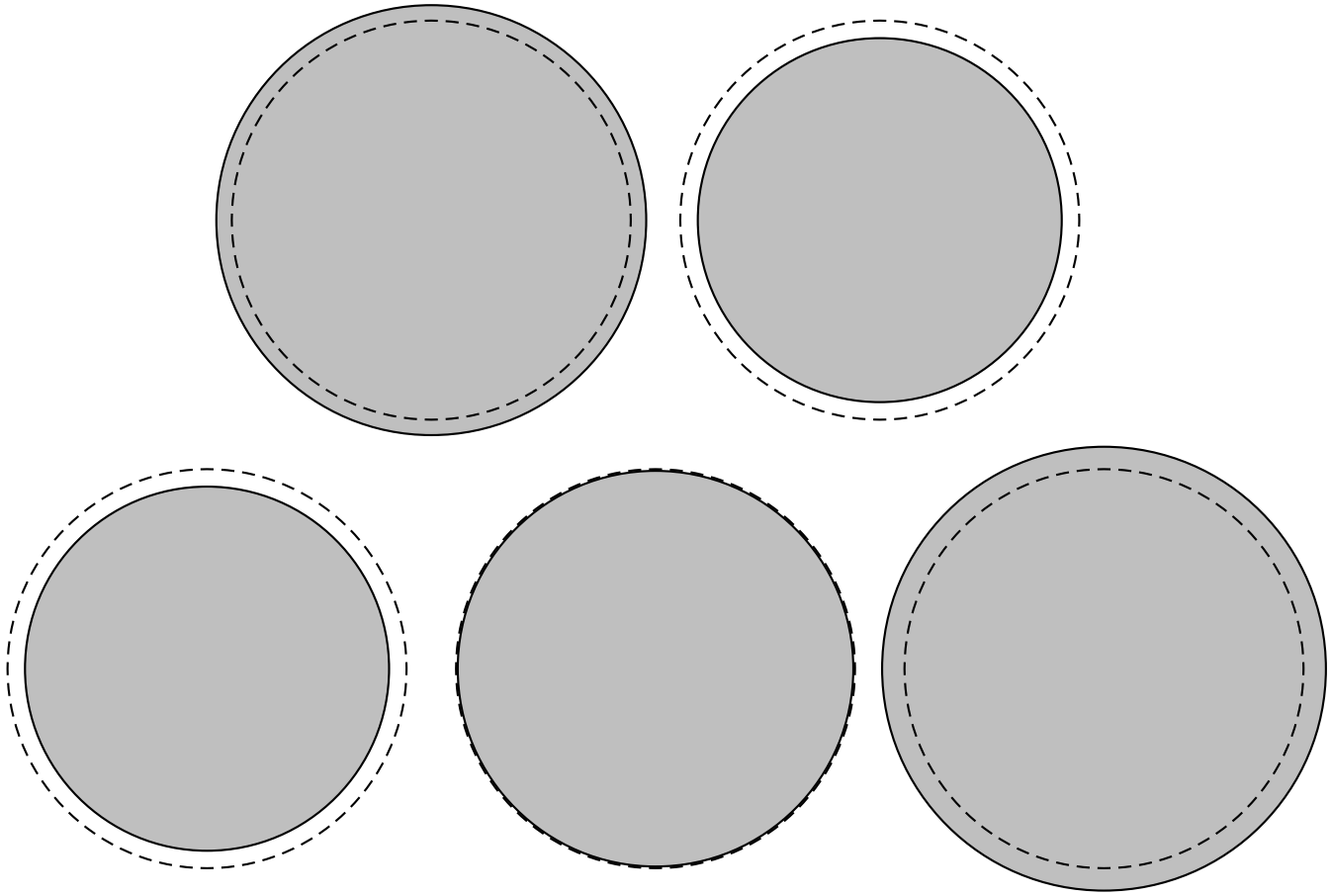
Thus

$$\begin{aligned}
P\{\tilde{\mathbf{n}} \in Q_i, \|\tilde{\mathbf{n}}\| \geq \rho_i\} &\leq P\{\tilde{\mathbf{n}} \notin Q_i, \|\tilde{\mathbf{n}}\| \leq \rho_i\} \\
P\{\tilde{\mathbf{r}} \in \widehat{R}_i | H_i\} &= P\{\tilde{\mathbf{n}} \in Q_i\} \\
&= P\{\tilde{\mathbf{n}} \in Q_i, \|\tilde{\mathbf{n}}\| \leq \rho_i\} + P\{\tilde{\mathbf{n}} \in Q_i, \|\tilde{\mathbf{n}}\| \geq \rho_i\} \\
&\leq P\{\tilde{\mathbf{n}} \in Q_i, \|\tilde{\mathbf{n}}\| \leq \rho_i\} + P\{\tilde{\mathbf{n}} \notin Q_i, \|\tilde{\mathbf{n}}\| \leq \rho_i\} \\
&= P\{\|\tilde{\mathbf{n}}\| \leq \rho_i\}
\end{aligned}$$

Thus

$$P\{\text{correct}, \tilde{\mathbf{r}} \in I_r\} \leq \sum_{i=0}^{M-1} P\{\|\tilde{\mathbf{n}}\| \leq \rho_i\} P\{H_i\}$$

In a similar manner we can upper bound the sum of the probabilities of the noise being in a sphere of radius ρ_i by the probability of a noise vector being in a sphere with the average volume. This is shown below. The gray filled in circles contain the exact area of each region. The dashed circle has area equal to the average area. the decoding



This is shown mathematically below.

Assume $P\{H_i\} = \frac{1}{M}$. Let $\overline{V} = \frac{1}{M}\sum_{i=0}^{M-1} V_i = \dfrac{V_r}{M}$ (i.e. the average volume of $V(\hat{R}_i)$). Let ρ be the radius of a sphere of volume V_r/M. Then

$$V_r/M = \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) d\tilde{\mathbf{n}} = \frac{1}{M}\sum_{i=0}^{M-1} V_i = \frac{1}{M}\sum_{i=0}^{M-1} \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) d\tilde{\mathbf{n}}$$

Now note that

$$\begin{aligned}
\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) d\tilde{\mathbf{n}} &= \frac{1}{M}\sum_{i=0}^{M-1} \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) d\tilde{\mathbf{n}} \\
&= \frac{1}{M}\sum_{i=0}^{M-1} \Bigg[\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) d\tilde{\mathbf{n}} \\
&\qquad + \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho_i\Big) d\tilde{\mathbf{n}}\Bigg].
\end{aligned}$$

Also

$$\begin{aligned}
\frac{1}{M}\sum_{i=0}^{M-1} \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) d\tilde{\mathbf{n}} &= \frac{1}{M}\sum_{i=0}^{M-1} \Bigg[\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) d\tilde{\mathbf{n}} \\
&\qquad + \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho\Big) d\tilde{\mathbf{n}}\Bigg].
\end{aligned}$$

Since the left side of the above 2 equations are identical, so must the right sides. Cancelling like terms we obtain

$$\frac{1}{M}\sum_{i=0}^{M-1}\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho_i\Big) d\tilde{\mathbf{n}} = \frac{1}{M}\sum_{i=0}^{M-1}\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho\Big) d\tilde{\mathbf{n}}. \qquad (\square)$$

Now

$$\begin{aligned}
\frac{1}{M}\sum_{i=0}^{M-1} P\Big\{\|\tilde{\mathbf{n}}\| \le \rho_i\Big\} &= \frac{1}{M}\sum_{i=0}^{M-1}\int I(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} \\
&= \frac{1}{M}\sum_{i=0}^{M-1}\Bigg[\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} \\
&\qquad + \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}}\Bigg].
\end{aligned}$$

Using the fact that $\|\tilde{\mathbf{n}}\| \ge \rho \Rightarrow p(\tilde{\mathbf{n}}) \le p\left(\frac{\rho}{\sqrt{N}} \cdot \underline{1}\right)$ in the second term gives

$$\begin{aligned}
\frac{1}{M}\sum_{i=0}^{M-1} P\Big\{\|\tilde{\mathbf{n}}\| \le \rho_i\Big\} &\le \frac{1}{M}\sum_{i=0}^{M-1}\Bigg[\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} \\
&\qquad + \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \ge \rho\Big) p\Big(\frac{\rho}{\sqrt{N}} \cdot \underline{1}\Big) d\tilde{\mathbf{n}}\Bigg].
\end{aligned}$$

Using ($\square$) we obtain

$$\begin{aligned}
\frac{1}{M}\sum_{i=1}^{M-1} P\{\|\tilde{\mathbf{n}}\| \le \rho_i\} &\le \frac{1}{M}\sum_{i=0}^{M-1}\Big[\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} \\
&\qquad + \int I\Big(\|\tilde{\mathbf{n}}\| \ge \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p\left(\frac{\rho}{\sqrt{N}} \cdot 1\right) d\tilde{\mathbf{n}}\Big].
\end{aligned}$$

Now using the fact that $\|\tilde{\mathbf{n}}\| \le \rho \Rightarrow p\left(\frac{\rho}{\sqrt{N}} \cdot 1\right) \le p(\tilde{\mathbf{n}})$ gives

$$\begin{aligned}
\frac{1}{M}\sum_{i=0}^{M-1} P\Big\{\|\tilde{\mathbf{n}}\| \le \rho_i\Big\} &\le \frac{1}{M}\sum_{i=0}^{M-1}\Bigg[\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} \\
&\qquad + \int I\Big(\|\tilde{\mathbf{n}}\| \ge \rho_i\Big) I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}}\Bigg] \\
&= \frac{1}{M}\sum_{i=0}^{M-1}\int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} \\
&= \int I\Big(\|\tilde{\mathbf{n}}\| \le \rho\Big) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} = P\Big\{\|\tilde{\mathbf{n}}\| \le \rho\Big\}.
\end{aligned}$$

Since

$$\overline{V} = \frac{V_r}{M} = B_N \rho^N \Rightarrow \rho = \left(\frac{V_r}{B_N M}\right)^{1/N}$$

$$P\{\text{correct}, \tilde{\mathbf{r}} \in I_r\} \le P\left\{\|\tilde{\mathbf{n}}\| \le \left(\frac{V_r}{B_N M}\right)^{1/N}\right\}.$$

Proof of Step 4:

$$\frac{V_r}{B_N M} = \frac{B_N\left(\sqrt{E + \frac{N_0}{2} + \Delta}\right)^N}{B_N M}$$

$$\begin{aligned}
\left(\frac{V_r}{B_N M}\right)^{1/N} &= \frac{\sqrt{E+\frac{N_0}{2}+\Delta}}{M^{1/N}} \\
P\left\{\|\tilde{\mathbf{n}}\| \le \left(\frac{V_r}{B_N M}\right)^{1/N}\right\} &= P\left\{\|\tilde{\mathbf{n}}\|^2 \le \left(\frac{1}{M}\right)^{2/N}\left(E+\frac{N_0}{2}+\Delta\right)\right\}.
\end{aligned}$$

Now it is easy to check that

$$\begin{aligned}
E\left[\|\tilde{\mathbf{n}}\|^2\right] &= \frac{N_0}{2} \\
\mathrm{Var}\left(\|\tilde{\mathbf{n}}\|^2\right) &= \frac{2}{N}\left(\frac{N_0}{2}\right)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
P\left\{\|\tilde{\mathbf{n}}\|^2 \right. &\le \left.\left(\frac{1}{M}\right)^{2/N}\left(E+\frac{N_0}{2}+\Delta\right)\right\} \\
= P\left\{\|\tilde{\mathbf{n}}\|^2 - \frac{N_0}{2} \right. &\le \left.\left(\frac{1}{M}\right)^{2/N}\left(E+\frac{N_0}{2}+\Delta\right) - \frac{N_0}{2}\right\} \\
\le P\left\{\left|\|\tilde{\mathbf{n}}\|^2 - \frac{N_0}{2}\right| \right. &\ge \left.\tilde{\Delta}\right\} \le \frac{\mathrm{Var}\left(\|\tilde{\mathbf{n}}\|^2\right)}{\tilde{\Delta}^2}
\end{aligned}$$

where

$$\tilde{\Delta} = -\left(\frac{1}{M}\right)^{2/N}\left(E+\frac{N_0}{2}+\Delta\right)+\frac{N_0}{2}.$$

So

$$P\left\{\|\tilde{\mathbf{n}}\|^2 \le \left(\frac{V_r}{B_N M}\right)^{2/N}\right\} \le \frac{2}{N}\left(\frac{N_0}{2}\right)^2\frac{1}{\tilde{\Delta}^2}.$$

Proof of Step 5: Assume $\frac{\log_2 M}{N} > \frac{1}{2}\log\left(1+\frac{2E}{N_0}\right)$. Then there exist a $\Delta > 0$ such that

$$\frac{\log_2 M}{N} > \frac{1}{2}\log_2\left(1+\frac{2E}{N_0}+\Delta/(N_0/2)\right)$$

or

$$M > \left(\frac{E+N_0/2+\Delta}{N_0/2}\right)^{N/2}$$

or

$$\tilde{\Delta} = \frac{N_0}{2} - \left(\frac{1}{M}\right)^{2/N}\left(E+\frac{N_0}{2}+\Delta\right) > 0$$

$$\tilde{\Delta} = \frac{N_0}{2} - \left(\frac{1}{M}\right)^{2/N}\left(E+\Delta+\frac{N_0}{2}\right)$$

$$\begin{aligned}
R &= \frac{\log_2 M}{N} \Rightarrow RN = \log_2 M \\
M &= 2^{RN} \Rightarrow \frac{1}{M} = 2^{-RN} \\
\tilde{\Delta} &= \frac{N_0}{2} - \left(2^{-RN}\right)^{\frac{2}{N}}\left(E+\Delta+\frac{N_0}{2}\right) \\
&= \frac{N_0}{2} - \left(2^{-2R}\right)\left(E+\Delta+\frac{N_0}{2}\right)
\end{aligned}$$

Thus

$$\widetilde{\Delta} \to \frac{N_0}{2} - 2^{-2R}\left(E + \Delta + \frac{N_0}{2}\right) > 0$$

$$\sqrt{N}\widetilde{\Delta} \to \infty \text{ as } N \to \infty$$

Proof of Step 6: Put steps 1-5 together.

Q.E.D.

Proof of (b):

$$\begin{aligned} \|\mathbf{s}_i\|^2 &\leq NE \\ \tilde{\mathbf{s}}_i &= \mathbf{s}_i/\sqrt{N} \Rightarrow \|\tilde{\mathbf{s}}_i\|^2 \leq E \end{aligned}$$

As in R_0 coding theorem choose $\tilde{\mathbf{s}}_i$ (thus $\mathbf{s}_i$) at random and average the error probability over all such $\tilde{\mathbf{s}}$. Since $\|\tilde{\mathbf{s}}_i\|^2 \leq E$ must have that

$$P\left\{\|\tilde{\mathbf{s}}_i\|^2 > E\right\} = 0$$

Let

$$p_{\tilde{\mathbf{s}}_i}(\alpha) = \begin{cases} \frac{1}{V_s}, & \|\alpha\|^2 \leq E \\ 0, & \|\alpha\|^2 > E \end{cases}$$

where V_s =volume of a sphere of radius $\sqrt{E}$

$$p_{\tilde{\mathbf{s}}_0,\tilde{\mathbf{s}}_1,\ldots,\tilde{\mathbf{s}}_{M-1}}(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}) = \prod_{i=0}^{M-1} p_{\tilde{\mathbf{s}}_i}(\alpha_i)$$

Then

$$\begin{aligned} P\left\{\|\tilde{\mathbf{s}}_i\| \leq \sqrt{E-\Delta}\right\} &= P\left\{\tilde{\mathbf{s}}_i \text{ in sphere of radius } \sqrt{E-\Delta}\right\} \\ &= \frac{\text{Volume of sphere of radius } \sqrt{E-\Delta}}{\text{Volume of sphere of radius } \sqrt{E}} \\ &= \left(\frac{E-\Delta}{E}\right)^{N/2} \\ &= \left(1 - \frac{\Delta}{E}\right)^{N/2} \to 0 \text{ as } N \to \infty. \end{aligned}$$

This implies that all points are located near the surface in a space of high dimensionality. Thus

$$P\{\text{error}\} = \sum_{i=0}^{M-1} P\{\text{error}|H_i\}\pi_i$$

$$P\{\text{error}|H_i\} = E_{\tilde{\mathbf{s}}_i,\tilde{\mathbf{n}}}\left[P\{\text{error}|H_i, \tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}\}\right]$$

Fix $\tilde{\mathbf{n}}, \tilde{\mathbf{s}}_i$, then an error will occur if any of the other $\tilde{\mathbf{s}}_j$, $j \neq i$ are in the shaded region. This is true since if $\tilde{\mathbf{s}}_k \in L$ then $d(\tilde{\mathbf{r}}, \tilde{\mathbf{s}}_k) \leq d(\tilde{\mathbf{r}}, \tilde{\mathbf{s}}_i)$ so that optimal receiver will make an error. Thus

$$P\{\tilde{\mathbf{s}}_j \in L \text{ for some } j \neq i | H_i, \tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}\} \leq \sum_{\substack{j=0 \\ j\neq i}}^{M} P\{\tilde{\mathbf{s}}_j \in L | H_i, \tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}\} \leq \frac{V_L(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}})}{V_s}$$

$$P\{\tilde{\mathbf{s}}_j \in L\} = (M-1)\frac{V_L(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}})}{V_s}$$

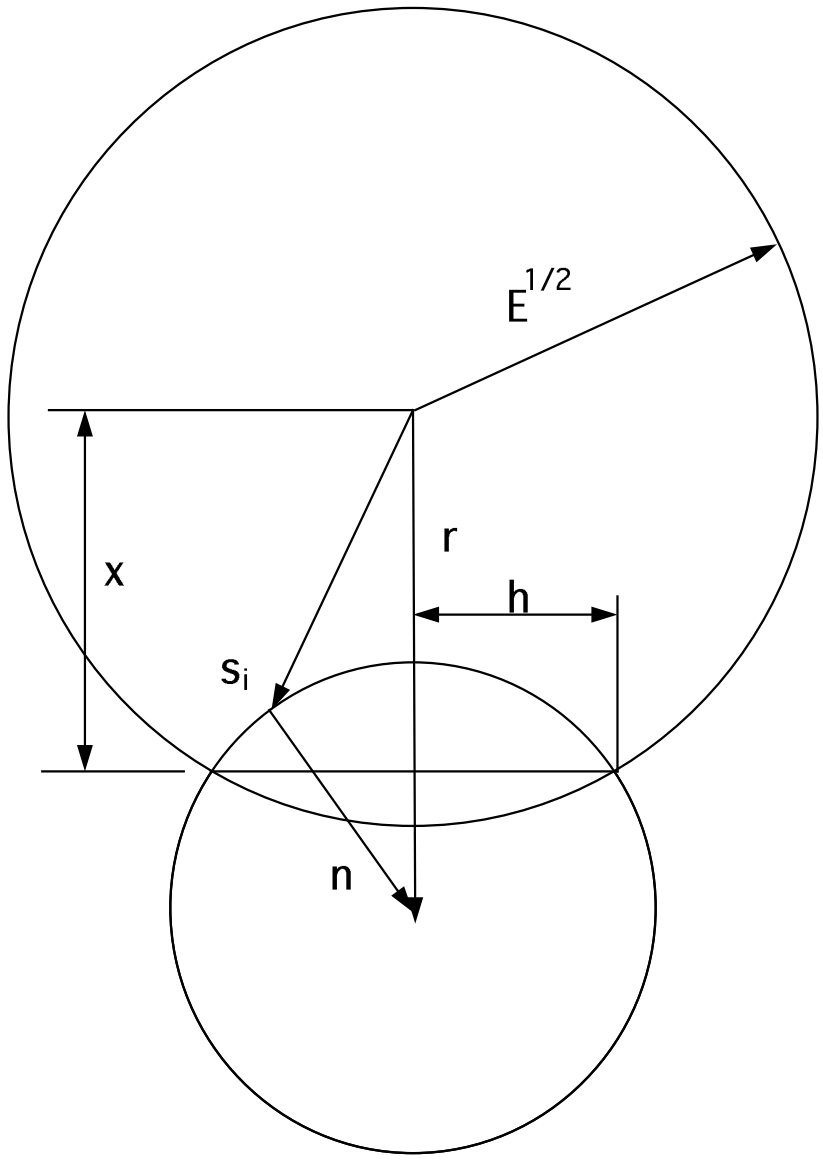


Figure 4.3: Capacity of Additive White Gaussian noise channels

Then by solving 2 equations in 2 unknowns (x and h)

$$x^2 + h^2 = E$$

and

$$(r-x)^2 + h^2 = n^2$$

we obtain

$$h^2 = E - \left(\frac{r^2 + E - n^2}{2r}\right)^2 \quad .$$

(We must have $r < n + \sqrt{E}$ for there to be a solution).
So

$$P\{\text{error}|H_i, \tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}\} \leq M \frac{V_L(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}})}{V_s}$$

Let

$$D = \left\{\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}} : \|\tilde{\mathbf{s}}_i\|^2 \geq E - \Delta, \|\tilde{\mathbf{n}}\|^2 \leq \frac{N_0}{2} + \Delta, \|\tilde{\mathbf{r}}\|^2 \geq \|\tilde{\mathbf{s}}_i\|^2 + \frac{N_0}{2} - \Delta\right\}.$$

Then

$$D^C = \left\{\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}} : \|\tilde{\mathbf{s}}_i\|^2 < E - \Delta \text{ or } \|\tilde{\mathbf{n}}\|^2 > \frac{N_0}{2} + \Delta \text{ or } \|\tilde{\mathbf{r}}\|^2 < \|\tilde{\mathbf{s}}_i\|^2 + \frac{N_0}{2} - \Delta\right\}.$$

Using the union bound yields

$$\begin{aligned} P\{D^C\} &\leq P\left\{\|\tilde{\mathbf{s}}_i\|^2 < E - \Delta\right\} + P\left\{\|\tilde{\mathbf{n}}\|^2 > \frac{N_0}{2} + \Delta\right\} + P\left\{\|\tilde{\mathbf{r}}\| < \|\tilde{\mathbf{s}}_i\|^2 + \frac{N_0}{2} - \Delta\right\} \\ &\leq \left(1 - \frac{\Delta}{E}\right)^{N/2} + \frac{2}{N}\left(\frac{N_0}{2}\right)^2 \frac{1}{\Delta^2} + P\left\{\|\tilde{\mathbf{r}}\|^2 \leq \|\tilde{\mathbf{s}}_i\|^2 + \frac{N_0}{2} - \Delta\right\} \end{aligned}$$

Consider the last term of the above.

$$\begin{aligned} P\left\{\|\tilde{\mathbf{r}}\|^2 \leq \|\tilde{\mathbf{s}}_i\|^2 + \frac{N_0}{2} - \Delta\right\} &= E\, P\left\{\|\tilde{\mathbf{r}}\|^2 \leq \|\alpha\|^2 + \frac{N_0}{2} - \Delta | \tilde{\mathbf{s}}_i = \alpha\right\} \\ &\leq E\left[\frac{4EN_0 + N_0^2}{N\Delta^2}\right] \\ &= \frac{4EN_0 + N_0^2}{N\Delta^2} \end{aligned}$$

$$\begin{aligned} \therefore P\{D^C\} &\leq \left(1 - \frac{\Delta}{E}\right)^{N/2} + \frac{2}{N}\left(\frac{N_0}{2}\right)^2 \frac{1}{\Delta^2} + \frac{4EN_0 + N_0^2}{N\Delta^2} = \frac{K}{N} \\ &\to 0 \text{ as } N \to \infty \\ P(D) &\to 1 \quad \text{but } P(D) \leq 1 \end{aligned}$$

$$\begin{aligned} P\{\text{error}|H_i\} &= \int P\{\text{error}|H_i, \tilde{\mathbf{s}}_i = \alpha, \tilde{\mathbf{n}} = \beta\} p_{\tilde{\mathbf{s}}_i}(\alpha) p_{\tilde{\mathbf{n}}}(\beta) d\alpha\, d\beta \\ &= \int_D P\{\text{error}|H_i, \tilde{\mathbf{s}}_i = \alpha, \tilde{\mathbf{n}} = \beta\} p_{\tilde{\mathbf{s}}_i}(\alpha) p_{\tilde{\mathbf{n}}}(\beta) d\alpha\, d\beta \\ &\quad + \int_{D^C} \underbrace{P\{\text{error}|H_i, \tilde{\mathbf{s}}_i = \alpha, \tilde{\mathbf{n}} = \beta\}}_{\leq 1} p_{\tilde{\mathbf{s}}_i}(\alpha) p_{\tilde{\mathbf{n}}}(\beta) d\alpha\, d\beta \\ &\leq \int_D P\{\text{error}|H_i, \tilde{\mathbf{s}}_i = \alpha, \tilde{\mathbf{n}} = \beta\} p_{\tilde{\mathbf{s}}_i}(\alpha) p_{\tilde{\mathbf{n}}}(\beta) d\alpha\, d\beta + P(D^C) \end{aligned}$$

If $\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}} \in D$ then

$$P\{\text{error}|H_i, \tilde{\mathbf{s}}_i = \alpha\ , \tilde{\mathbf{n}} = \beta\ \} \leq M \frac{V_L(\alpha\ , \beta\ )}{V_s}$$

We upper bound the volume V_L by the volume of sphere of radius h, i.e.

$$V_L(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}) \leq B_N(h(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}))^N \leq B_N(h^*)^N$$

where h^* is maximum radius over all $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}) \in D$. For $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}) \in D$ we have that

$$\begin{aligned}
\|\tilde{\mathbf{s}}_i\|^2 &\geq E - \Delta \\
n^2 \triangleq \|\tilde{\mathbf{n}}\|^2 &\leq \frac{N_0}{2} + \Delta \\
r^2 \triangleq \|\tilde{\mathbf{r}}\|^2 &= \|\tilde{\mathbf{s}}_i + \tilde{\mathbf{n}}\|^2 \geq \|\tilde{\mathbf{s}}_i\|^2 + \frac{N_0}{2} - \Delta \geq E + \frac{N_0}{2} - 2\Delta.
\end{aligned}$$

Let

$$h^2 \triangleq E - \left(\frac{r^2 + E - n^2}{2r}\right)^2$$

Claim: For $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}},) \in D$, h is an increasing function of n and a decreasing function of r.
Proof of claim:
First we will show that $h = E - \left(\frac{r^2+E-n^2}{2r}\right)^2$ is nonnegative for $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}},) \in D$. We need to show that

$$\begin{aligned}
\sqrt{E} &> \frac{r^2 + E - n^2}{2r} \\
2r\sqrt{E} &> r^2 + E - n^2 \\
0 &> r^2 - 2r\sqrt{E} + E - n^2 \\
n^2 &> \left(r - \sqrt{E}\right)^2 \\
n &> r - \sqrt{E} \\
r &< n + \sqrt{E}
\end{aligned}$$

This last inequality must be true by the triangle inequality since $r \leq ||\tilde{\mathbf{s}}_i|| + n < \sqrt{E} + n$.

We now prove that h has the stated properties for $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}},) \in D$ Let

$$z = h^2 = E - \left(\frac{r^2 + E - n^2}{2r}\right)^2$$

then we must show that $\dfrac{\partial z}{\partial r} < 0$. Now

$$\begin{aligned}
\frac{\partial z}{\partial r} &= -\left(\frac{r^2 + E - n^2}{2r}\right)\left(\frac{1}{2} - \frac{E - n^2}{2r^2}\right) \\
&= -\left(\frac{r^2 + E - n^2}{2r}\right)\left(\frac{r^2 - E + n^2}{2r^2}\right) \\
&= \frac{-1}{4r^3}\left(r^2 + (E - n^2)\right)\left(r^2 - (E - n^2)\right) \\
&= \frac{-1}{4r^3}\left(r^4 - (E - n^2)^2\right) \\
&\leq 0 \text{ if } r^4 > (E - n^2)^2
\end{aligned}$$

Thus if $r^2 > |E - n^2|$ then $\frac{\partial z}{\partial r} < 0$. Since for $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}},) \in D$

$$r^2 \geq E + \frac{N_0}{2} - 2\Delta \text{ and } E \geq E - n^2$$

It is clear that $r^2 > |E - n^2|$ if $\Delta \leq \frac{N_0}{4}$. Now consider $\frac{\partial z}{\partial n}$.

$$\frac{\partial z}{\partial n} = -2\left(\frac{r^2 + E - n^2}{2r}\right)\left(-\frac{n}{r}\right) = \frac{2n}{r^2}(r^2 + E - n^2)$$

Then

$$\frac{\partial z}{\partial n} > 0 \text{ if } r^2 + E > n^2$$

Since for $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}},) \in D$

$$r^2 + E \geq E + \frac{N_0}{2} - 2\Delta + E \text{ and } \frac{N_0}{2} + \Delta \geq n^2$$

$$\frac{\partial z}{\partial n} > 0 \text{ if } \Delta < \frac{2E}{3}$$

Thus h is a decreasing function of r and an increasing function of n for $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}},) \in D$ if $\Delta < \min\{2E/3, N_0/4\}$. Thus to maximize h over $(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}},) \in D$ we need to put the largest value for n and the smallest value for r, i.e.

$$\begin{aligned}
h^2 &\leq E - \left(\frac{r_{\min}^2 + E - n_{\max}^2}{2r_{\min}}\right)^2 \\
&= E - \left(\frac{E + N_0/2 - 2\Delta + E - \frac{N_0}{2} - \Delta}{2\sqrt{E + \frac{N_0}{2} - 2\Delta)}}\right)^2 \\
&= \frac{4E\left(E + \frac{N_0}{2} - 2\Delta\right) - (2E - 3\Delta)^2}{4\left(E + \frac{N_0}{2} - 2\Delta\right)} \\
&= \frac{4E^2 + 4EN_0/2 - 8E\Delta - (4E^2 - 12E\Delta + 9\Delta^2)}{4\left(E + \frac{N_0}{2} - 2\Delta\right)} \\
h^2 &\leq \frac{EN_0/2 + E\Delta - 9/4\Delta^2}{E + \frac{N_0}{2} - 2\Delta} \\
h^2 &\leq \left(\frac{EN_0/2}{E + N_0/2}\right) + \Delta K(E, N_0) \\
h &\leq \sqrt{\frac{EN_0/2}{E + N_0/2}} + \sqrt{\Delta}\ K^{1/2}(E, N_0) \\
h_{\max} &\leq \sqrt{\frac{EN_0/2}{E + N_0/2}} + \sqrt{\Delta}\ K^{1/2}(E, N_0) \\
\frac{V_L}{V_S} &\leq \frac{\left(\sqrt{\frac{EN_0/2}{E+N_0/2}} + \delta\right)^N}{\left(\sqrt{E}\right)^N} \\
M\frac{V_L^*}{V_S} &= \left(\sqrt{\frac{1}{1 + 2E/N_0}} + \frac{\delta}{\sqrt{E}}\right)^N M
\end{aligned}$$

$$\begin{aligned}
P\{\text{error}|H_i\} &\leq \int_D \frac{MV_L^*}{V_S}\ p(\tilde{\mathbf{s}}_i, \tilde{\mathbf{n}}) d\tilde{\mathbf{s}}_i d\tilde{\mathbf{n}} + P(D^C) \\
&\leq \left(\sqrt{\frac{1}{1 + 2E/N_0}} + \tilde{\delta}\right)^N MP(D) + P(D^C) \\
&\leq 2^{N\left(R - \frac{1}{2}\log_2\left(1 + 2E/N_0 + \delta/\sqrt{E}\right)\right)} + \frac{K}{N}
\end{aligned}$$

$$R = \frac{\log_2 M}{N}$$

Thus if

$$R < C = \frac{1}{2}\log_2\left(1 + \frac{2E}{N_0}\right)$$

then

$$P_e \to 0 \text{ as } N \to \infty$$

(Since if $R < C$ $\exists$ a small Δ such that δ is small enough so that $R < \frac{1}{2}\log_2(1+\frac{2E}{N_0})$.)

Q.E.D.

2. Capacity of Discrete Memoryless Channels:

Consider now a transmitter that uses signals for which the orthonormal expansion has coefficients that come from a finite alphabet. Also, the receiver correlates the received signal with the orthonormal signals and quantizes each of the outputs into a finite number of levels. Furthermore assume that each dimension is quantized in an identical fashion. Then the channel can be modeled by a discrete memoryless channel, i.e. the sequence of random variables representing the output of the quantizers is an independently distributed sequence conditioned on the input sequence.

Let A be the input alphabet and B be the output alphabet. Let $p(b|a)$ be the transition probability from input a to output b. An (n,M) code (or signal) is a set of M vectors of length n with each component taking values in A. The rate of an (n,M) code is $\log_2 M/n$ bits per dimension or bits per channel use. Let $C \triangleq \max_X I(X;Y)$ where

$$I(X;Y) = \sum_{x,y} p(x,y)\log_2 \frac{p(y|x)}{p(y)}$$

and the maximum is over all distributions on the random variable X.

Channel Coding Theorem for DMC:

(a) There exists codes of rate R with n sufficiently large and with the probability of making an error at the decoder arbitrarily small if $R < C$.

(b) Furthermore for any $R > C$ every code has error probability bounded away from 0.

We will prove (a):

Let $q(x)$ be a distribution such that

$$C = \sum_{x,y} p(y|x)q(x)\log_2 \frac{p(y|x)}{\sum_{x'} p(y|x')q(x')}.$$

Let $R < R' < C$. Let x and y be random variables with x having distribution $q(x)$ and y having distribution $\sum_x p(y|x)q(x)$. Consider an (n,M) code with codewords (signals) $\{\mathbf{x}_1, \mathbf{x}_2, ...\mathbf{x}_M\}$ with $\mathbf{x}_i = (x_{i,1}, ..., x_{i,n})$. Let $I(x;y) = \log_2 \frac{p(y|x)}{p(y)}$. Let

$$I(\mathbf{x};\mathbf{y}) = \log_2 \frac{p(\mathbf{y}|\mathbf{x})}{p(\mathbf{y})} = \sum_{i=1}^{n} \log_2 \frac{p(y_i|x_i)}{p(y_i)}$$

since for a memoryless channel

$$p(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{n} p(y_i|x_i).$$

Let $S(\mathbf{y}) = \{\mathbf{x} : I(\mathbf{x};\mathbf{y}) \geq nR'\}$. The decoder we consider (which is suboptimal for any finite n) decides $\mathbf{x}_i$ was transmitted if the received vector is $\mathbf{y}$ and

$$\mathbf{x}_j \in S(\mathbf{y}) \text{ is true for } j = i \textit{ only}$$

Thus the probability of error given $\mathbf{x}_i$ transmitted is

$$P_{e,i} = P\{\mathbf{x}_i \notin S(\mathbf{y})|H_i\} + \sum_{j=0, j\neq i}^{M-1} P\{\mathbf{x}_j \in S(\mathbf{y})|H_i\}.$$

The first term of the above is

$$P\{\mathbf{x}_i \notin S(\mathbf{y})|H_i\} = P\{I(\mathbf{x}_i;\mathbf{y}) < nR'|H_i\}.$$

As usual we will consider the set of all possible codes and average the error probability over such a collection. Chose the distribution of the codewords with an i.i.d. distribution on the components of each codeword with each component having distribution $q(x)$, i.e.

$$P\{\mathbf{x} = \mathbf{x}_i\} = \prod_{j=1}^{n} q(x_{i,j}).$$

We now want to obtain bounds on the error probability. The first term in the error probability can be shown to go to zero for larger n by the weak law of large numbers as follows.

$$I(\mathbf{x}_i;\mathbf{y}) = \sum_{j=1}^{n} I(x_{i,j};y_i)$$

$$E\left[I(\mathbf{x}_i;\mathbf{y})|H_i\right] = nI(X;Y) = nC$$

Thus

$$P\{I(\mathbf{x}_i;\mathbf{y}) < nR'|H_i\} = P\{\sum_{j=1}^{n} I(x_{i,j};y_j) - nI(X;Y) < n(R'-C)|H_i\}.$$

Since $R' < C$ this is the probability that a random variable is smaller than it's mean by a positive quantity. For n large this must go to zero as n goes to ∞.

Now consider the second term in the error probability.

$$\begin{aligned}
\sum_{j=1,j\neq i}^{M} P\{\mathbf{x}_j \in S(\mathbf{y})|H_i\} &= \sum_{j\neq i}\sum_{\mathbf{x}_1}\cdots\sum_{\mathbf{x}_M}\sum_{\mathbf{y},I(\mathbf{x}_j;\mathbf{y})\geq nR'} p(\mathbf{x}_1)\ldots p(\mathbf{x}_M)p(\mathbf{y}|\mathbf{x}_i) \\
&= \sum_{j\neq i}\sum_{\mathbf{x}_i}\sum_{\mathbf{x}_j}\sum_{\mathbf{y},I(\mathbf{x}_j;\mathbf{y})\geq nR'} p(\mathbf{x}_i)p(\mathbf{x}_j)p(\mathbf{y}|\mathbf{x}_i) \\
&= \sum_{j\neq i}\sum_{\mathbf{x}_j}\sum_{\mathbf{y},I(\mathbf{x}_j;\mathbf{y})\geq nR'} p(\mathbf{x}_j)p(\mathbf{y})
\end{aligned}$$

Now

$$\begin{aligned}
I(\mathbf{x}_j;\mathbf{y}) \geq nR' \quad &\Rightarrow \quad \log\frac{p(\mathbf{y}|\mathbf{x}_j)}{p(\mathbf{y})} \geq nR' \\
&\Rightarrow \quad \frac{p(\mathbf{y}|\mathbf{x}_j)}{p(\mathbf{y})} \geq 2^{nR'} \\
&\Rightarrow \quad \frac{p(\mathbf{y},\mathbf{x}_j)}{p(\mathbf{y})p(\mathbf{x}_j)} \geq 2^{nR'} \\
&\Rightarrow \quad p(\mathbf{y})p(\mathbf{x}_j) \leq p(\mathbf{y},\mathbf{x}_j)2^{-nR'}
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{j\neq i}\sum_{\mathbf{x}_j}\sum_{\mathbf{y},I(\mathbf{x}_j;\mathbf{y})\geq nR'} p(\mathbf{x}_j)p(\mathbf{y}) &\leq \sum_{j\neq i}\sum_{\mathbf{x}_j}\sum_{\mathbf{y},I(\mathbf{x}_j;\mathbf{y})\geq nR'} p(\mathbf{x}_j,\mathbf{y})2^{-nR'} \\
&\leq \sum_{j\neq i} 2^{-nR'} \\
&\leq M2^{-nR'} \\
&\leq 2^{-n(R'-R)}.
\end{aligned}$$

Since $R < R'$ this term also goes to zero as n goes to ∞. Thus we have shown that as n goes to ∞ the error probability given $\mathbf{x}_i$ was transmitted goes to zero if the rate is less than the capacity. This however does not imply that there

exist a code $\mathbf{x}_1, ..., \mathbf{x}_M$ such that $P_{e,1}, ..., P_{e,M}$ are simlutaeously small. It is possible that $P_{e,i}$ is small for some code for which $P_{e,j}$ is large. We now show that we can simultaneously make each of the error probabilities small simultaneously. First chose a code with $M = 22^{Rn}$ codewords for which the average error probability is less than say $\varepsilon_n/2$ for large n. If more than 2^{nR} of these codewords has $P_{e,i} \geq \varepsilon_n$ then the average error probability would be greater than $\varepsilon_n/2$, a contradiction. Thus at least $M/2 = 2^{nR}$ of the codewords must have $P_{e,i} \leq \varepsilon_n$. So delete the codewords that have $P_{e,i} \geq \varepsilon_n$ (less than half). We obtain a code with (at least) 2^{nR} codewords with $P_{e,i} \to 0$ as $n \to \infty$ for $R < C$.

3. Summary of Channel Models, Capacity and Cutoff Rate

Let us summarize the various channels and the capacity and cutoff rate for each channel.

Additive White Gaussian noise channel

$$C = \frac{1}{2}\log_2(1 + \frac{2E}{N_0})\text{bits per dimension}$$

$$R_0 = \frac{\log_2 e}{2}\left[1 + \frac{E}{N_0} - \sqrt{1 + (\frac{E}{N_0})^2}\right] + \frac{1}{2}\log_2\left[\frac{1}{2}(1 + \sqrt{1 + (\frac{E}{N_0})^2}\right]$$

where E is the energy per dimension of the signal set.

If we restrict ourselves to binary input, i.e. the coefficient of each orthonormal signal is either $+\sqrt{E}$ or $-\sqrt{E}$ then (for equally likely inputs)

$$\begin{aligned} R < C &= \int_{y=-\infty}^{\infty} p(y|x)\log_2\left(\frac{p(y|x)}{p(y)}\right)dy \\ &= 1 - \int_{y=-\infty}^{\infty} p(y|x)\log_2\left(1 + \frac{p(y|-1)}{p(y|+1)}\right)dy \\ &= 1 - \int_{-\infty}^{\infty} g(y-\beta)\log_2\left[1 + \exp\{-2y\beta\}\right]dy \end{aligned}$$

where $\beta = \sqrt{2E/N_0}$ and $g(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$. The cutoff rate in this case is

$$R_0 = 1 - \log_2(1 + e^{-E/N_0}).$$

For a discrete memoryless channel

$$C = \max_X I(X;Y)$$

where the maximization is over all random variables on the input to the channel.

For a communication systems using BPSK modulation with hard decision decoding the input and output alphabets are $\{\pm 1\}$ and corresponds to a discrete memoryless channel. The transition probabilities are

x	y	$p(y\|x)$
$+1$	$+1$	$1 - Q(\sqrt{(\frac{2E}{N_0})})$
$+1$	-1	$Q(\sqrt{(\frac{2E}{N_0})})$
-1	$+1$	$Q(\sqrt{(\frac{2E}{N_0})})$
-1	-1	$1 - Q(\sqrt{(\frac{2E}{N_0})})$

The capacity of this channel is

$$C = 1 - H_2(Q(\sqrt{\frac{2E}{N_0}}))$$

where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the binary entropy function.

The cutoff rate in this case is

$$R_0 = \max_{X_1,X_2} -\log_2 E[J(X_1,X_2)]$$

where the maximization is over all independent identically distributed random variables X_1 and X_2 and where

$$J(x_1,x_2) = \sum_{y\in B} \sqrt{p(y|x_1)p(y|x_2)}.$$

Usually a communication engineer is not as interested in E/N_0 as he/she is in the energy needed to transmit an information bit. These are however related by the rate of transmission.

$$E_b = E/R$$

The coding theorem for additive white Gaussian noise shows that signals with low error probability with E joules per dimension exist provide the number of bits per dimension R satisfies

$$R < \frac{1}{2}\log_2(1+\frac{2E}{N_0})$$

Thes can be rewritten as follows

$$\begin{aligned} 2^{2R} &< 1+\frac{2E}{N_0} \\ \frac{2^{2R}-1}{2} &< \frac{E}{N_0} \\ \frac{E}{N_0} &> \frac{2^{2R}-1}{2} \\ \frac{E_b}{N_0} &= \frac{E}{N_0 R} > \frac{2^{2R}-1}{2R} \end{aligned}$$

So another interpretation is that signals with R bits per dimension exist if the ratio of E_b to N_0 is larger than $\frac{2^{2R}-1}{2R}$.

For low rates the signal-to-noise ratio needed for reliable communication is found by taking the limit as $R \to 0$. Using L'Hospitals rule we obtain

$$\lim_{R\to 0} \frac{2^{2R}-1}{2R} = \lim_{R\to 0} 2^{2R}\ln 2 = \ln 2 = -1.59\text{dB}.$$

Problem: Show that for the binary input channel at low rates (and low E/N_0) the required E_b/N_0 is -1.59 dB. Hint: Do a Taylor series expansion of C in terms of E/N_0.

1. Capacity of nonwhite channels

Consider a channel with noise which has power spectral density $N(f)$. Assume that the noise is not white. Let P be the input power to the channel. Then the capacity of the channel is given by

$$C = \int_G \frac{1}{2}\log_2\left[\frac{B}{N(f)}\right] df$$

where G is the set of frequencies for which $N(f) \le B$ and B is the solution to

$$P = \int_G [B - N(f)]\, df$$

Cutoff rate of mismatched channels. Cutoff rate for AWGN (Shannon). Relation between bandwidth and dimensions.

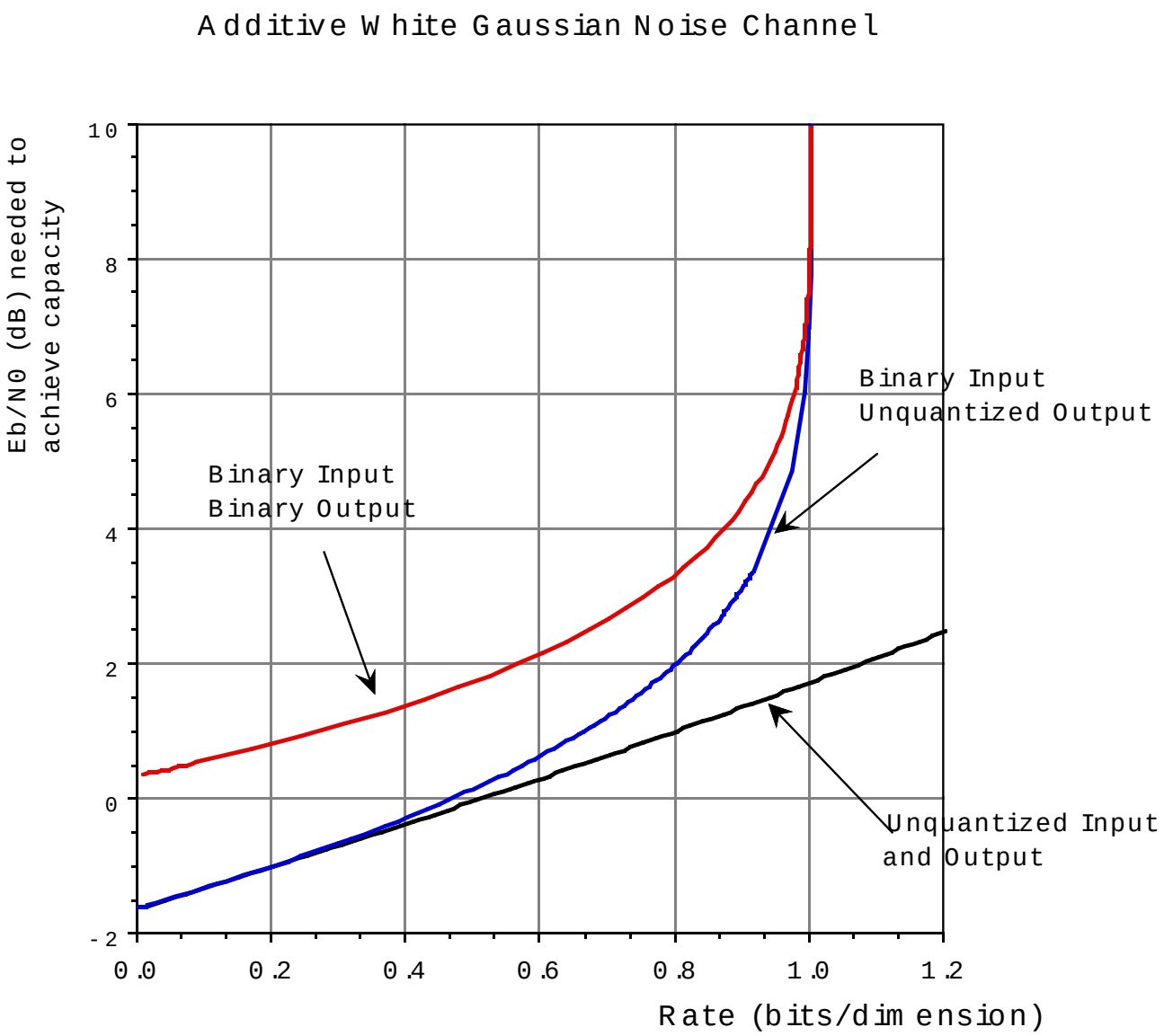


Figure 4.4: Capacity of Additive White Gaussian noise channels

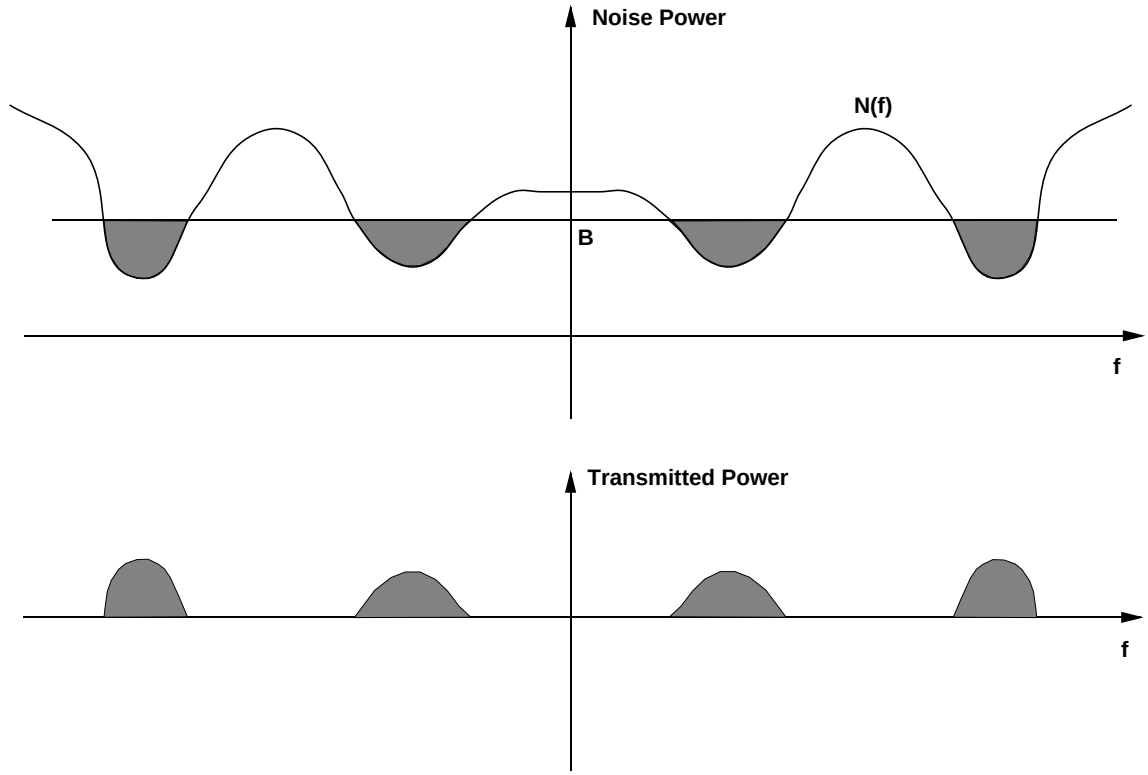


Figure 4.5: Water filling argument for Nonwhite Noise Channels

2. Comments

We showed that there exist a code for which the error probability goes to zero as N goes to ∞ provided the rate is less than the capacity. The convergence of the error probability was shown to be faster than $1/N$. In fact, the convergence of the error probability to zero is exponential.

Fact: There exist a set of M signals (codes) in N dimensions where the error probability is upper bounded by

$$P_e \leq 2^{-N(E(R))}$$

where $E(R)$ is called the error exponent and is positive for all rates less than capacity.

$$E(R) > 0, \;\; 0 \leq R < C.$$

Using the Gallager bound and averaging over all possible signal sets one can derive an error exponent. An improved error exponent for rates less than some critical rate is possible by excluding (expurgating) certain bad codewords from the averaging process. For high rates (above the critical rate but below capacity the bound can not be improved. There is a lower bound t=on the error probability (known as the sphere packing bound) which is identical to the upper bound for rates above the critical rate. The Union-Bhattacharyya bound gives a linear error exponent. That is

$$P_e \leq 2^{N(E(R)}$$

where $E(R) = R_0 - R$. The largest number R_0 such that there is such a bound is known as the cutoff rate. Because the Gallager bound is identical to the lower bound and is identical to the U-B cbound at the critical rate the cutoff rate is the parameter determined from the U-B bound

4. Bandwidth, Time and Dimensionality

In a bandwidth of W Hz and time duration of T seconds there are $N = 2WT$ orthogonal dimensions. We can recaluclate the capacity in terms of the largest rate in bits per second as follows. Since there are $2W$ "useable" dimensions every second the capacity in bits per second becomes

$$\mathbf{C} = 2WC = 2W\frac{1}{2}\log_2(1+\frac{2E}{N_0})$$

Notice also that

$$E \text{ (joules/dimension)} = \frac{P \text{ (joules/sec)}}{2W\text{(dimensions/sec)}}$$

Thus the capacity is

$$\mathbf{C} = 2WC = W\log_2(1+\frac{P}{N_0 W})$$

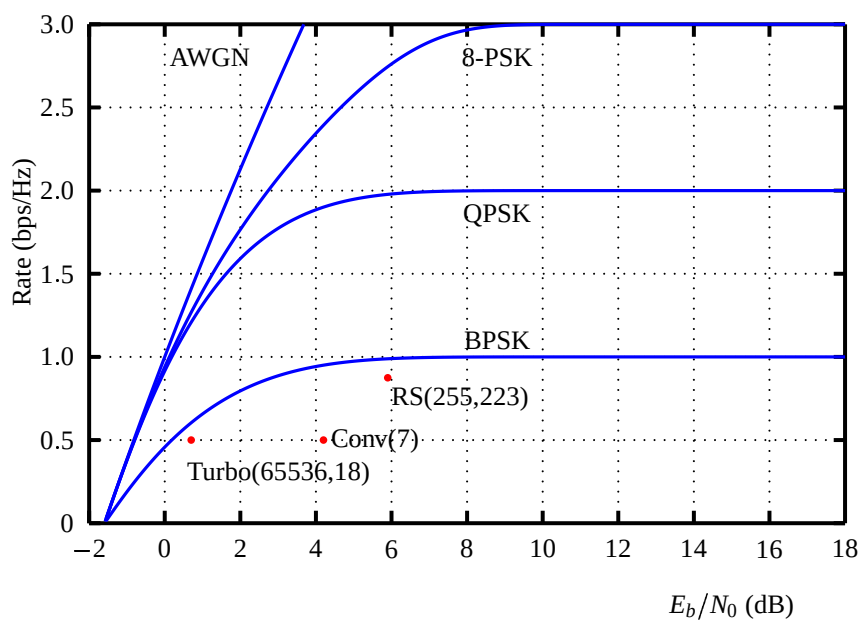


Figure 4.6: Transmission Rate versus Signal-to-Noise Ratio

5. Fundamental limits with nonzero error probability

Another interesting question is what are the fundamental limits if the error probability is not required to be zero. In this case we must combine the theory for source coding with that of channel coding/modulation. If we start witha source producing data at a rate of R_s bits/second with probability 1/2 of the data being 0 and probability of 1/2 of the data being 1 then we can compress that data source into a source with a smaller rate by allowing some errors in the reproduction. If we require the error probability to be $P_{e,b}$ or less then we can compress the data to the rate

$$\begin{aligned} R_1 &= R_s(1-H_2(P_{e,b})) \\ &= R_s(1+P_{e,b}\log_2(P_{e,b})+(1-P_{e,b})\log_2(1-P_{e,b})) \end{aligned}$$

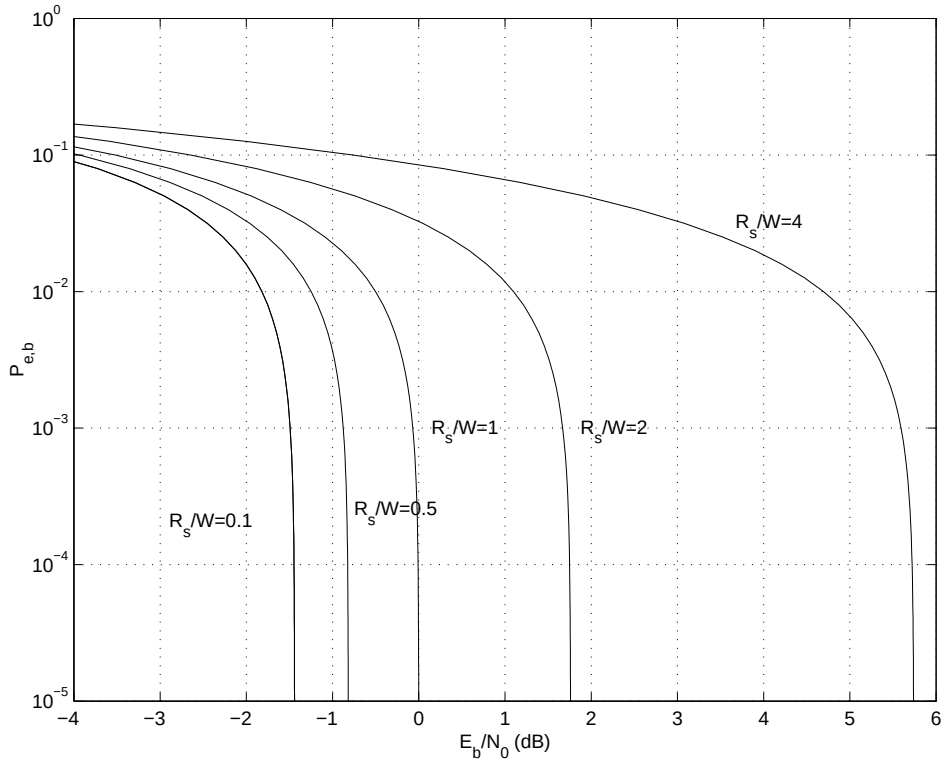


Figure 4.7: Funamental limit on error probability and signal-to-noise ratio for different bandwidth efficiencies

These compressed bits can be reliably communicated over a channel with bandwidth W Hz using power P watts provided

$$R_1 < W \log_2(1 + \frac{P}{N_0 W})$$

Thus error probability $P_{e,b}$ is possible provided

$$R_s(1 + P_{e,b}\log_2(P_{e,b}) + (1 - P_{e,b})\log_2(1 - P_{e,b})) < W \log_2(1 + \frac{P}{N_0 W})$$

We can express this in terms of energy per information bit by writing $E_b = P/R_s$. Using this relation the condition for achieving error probability $P_{e,b}$ is

$$1 + P_{e,b}\log_2(P_{e,b}) + (1 - P_{e,b})\log_2(1 - P_{e,b}) < \frac{W}{R_s}\log_2(1 + \frac{E_b}{N_0}\frac{R_s}{W})$$

6. Problems

1. In this problem we are considering a communications system operating in the presence of white Gaussian noise with power spectral density $N_0/2$. Each signal has energy EN where N is the number of dimensions and E is the energy per dimension. Let E_b be the energy per information bit. (a) If M signals are used in N dimensions with $M = 2^{N/2}$ and N and M very large what is the minimum value of E_b/N_0 such that the error probability can be made arbitrarily small (by making M and N very large) for the best choice of signals? (b) If M signals are used in N dimensions with $M = N$ and N and M very large what is the minimum value of E_b/N_0 such that the error probability can be made arbitrarily small (by making M and N very large) for the best choice of signals? (c) What set of signals with $M = N$ has error probability arbitrarily small (by making M and N very large) if E_b/N_0 is greater than the answer found in part (b) ?

2. In this problem we are considering a communications system operating in the presence of white Gaussian noise with power spectral density $N_0/2$. Each signal has energy no greater than EN where N is the number of dimensions and E is the energy per dimension. Let E_b be the energy per information bit.

   (a) If M signals are used in N dimensions with $M = 2^{N/2}$ and N and M very large what is the minimum value of E_b/N_0 such that the error probability can be made arbitrarily small (by making M and N very large) for the best choice of signals?

   (b) If M signals are used in N dimensions with $M = N$ and N and M very large what is the minimum value of E_b/N_0 such that the error probability can be made arbitrarily small (by making M and N very large) for the best choice of signals?

   (c) What set of signals with $M = N$ has error probability arbitrarily small (by making M and N very large) if E_b/N_0 is greater than the answer found in part (b) ?

3. Using the formula for the cutoff rate (extended to continuous alphabets by replacing sums by integrals where appropriate) calculate the cutoff rate of a binary input, continuous output Gaussian channel. (Binary input means that each coefficient in the orthogonal representation of the signal has two possible values $(+\sqrt{E}, -\sqrt{E})$.

4. An optical communication system transmits one of M equally likely symbols by pulsing a laser during one of M (nonoverlapping) time slots. That is to transmit symbol i, $i = 0, 1, \ldots, M-1$ we turn a laser (pointed at the receiver) on during the time slot $[iT/M,\ (i+1)T/M)$. During each time slot the receiver counts the number of photons received. Let N_i be the number of photons received during slot i, $i = 0, 1, \ldots, M-1$. Then if symbol i is transmitted it is known that the number of photons detected in slot j, N_j, has a Poisson distribution; i.e.

$$P\{N_j = l | \text{symbol } i \text{ transmitted}\} = \begin{cases} \frac{\lambda_1^l e^{-\lambda_1}}{l!} & j = i \\ \frac{\lambda_0^l e^{-\lambda_0}}{l!} & j \neq i \end{cases}$$

   Furthermore N_i, $i = 0, 1, \ldots, M-1$ are statistically independent. Calculate the cutoff rate for this channel.

5. It is known that the capacity of a certain binary input non Gaussian channel is given by

$$C = \begin{cases} 0 & E/N_0 < K \\ 1 - \frac{K}{E/N_0} & E/N_0 \geq K \end{cases}$$

   in information bits/channel use (or information bits/dimension) where K is a constant, $N_0/2$ is the average power of the noise and E is the energy/channel use.

   (a) Determine (explicitly) the minimum value of E_b/N_0 necessary for reliable communications (arbitrarily small error probability) when using a code of rate R information bits/channel use.

   (b) Determine the code rate that minimizes the answer found in part (a). (i.e. if you were going to design a code for this channel what code rate would you suggest to use?)

   (c) Determine the capacity in bits/sec. of the channel if there is an absolute bandwidth of W Hz. available.

6. For any channel let $(E_b/N_0)_{min}$ be the minimum value of E_b/N_0 such that reliable communications is possible.

   (a) Consider the AWGN channel. If we restrict the class of codes to have a binary alphabet what is the loss in $(E_b/N_0)_{min}$ compared to allowing nonbinary codes at low rates (close to zero)?

   (b) If we use a hard decision device on the output of the channel what is the loss in $(E_b/N_0)_{min}$ for binary codes as compared to binary codes with soft decisions at low code rates (close to zero)?

7. Compare the capacity of a binary input, binary output channel formed from an additive Gaussian noise channel to the capacity of a unquantized input, unquantized output channel at very low signal-to-noise ratios (or low rates). Use a Taylor series expansion of the capacity of the former to show that the loss in hard decision quantization is $\pi/2$ or 1.96dB.

8. Compare the cutoff rate of a binary input, binary output channel formed from an additive Gaussian noise channel to the cutoff rate of an binary input, unquantized output channel at very low signal-to-noise ratios (or low rates). Show that the loss in hard decision quantization is $\pi/2$ or 1.96dB.

9. (a) A communication system transmits one of M orthogonal signals. The receiver forms decision variables $Y_1, ..., Y_M$ with the property that given signal i transmitted the joint density function of the output factors as

$$p(y_1, ..., y_M | H_i) = p_1(y_i) \prod_{j \neq i} p_0(y_j)$$

where $p_1(y)$ corresponds to a density function with signal present and $p_0(y)$ corresponds to a density function with signal absent. Show that the capacity (in information symbols/channel symbols) can be expressed as

$$C = 1 - \int_{y_1, ..., y_M} \log_M \left[1 + \sum_{j=2}^{M} \frac{p(y_1, ..., y_M | H_j)}{p(y_1, ..., y_M | H_0)} \right] dy_1 \cdots dy_{M-1}$$

(b) Show that the cutoff rate for such a channel is given by

$$R_0 = 1 - \log_M [1 + (M-1)D]$$

where

$$D = \left[\int_{-\infty}^{\infty} \sqrt{p(y_1|H_1) p(y_1|H_2)} dy_1 \right]^2$$

10. A communication system uses a N dimensions and a code rate of R bits/dimension. The goal is not low error probability but high throughput (expected number of successfully received information bits per coded bit in a block on length N). If we use a low code rate then we have high success probability for a packet but few information bits. If we use a high code rate then we have a low success probability but a larger number of bits transmitted. Assume the channel is an additive white Gaussian noise channel and the input is restricted to binary modulation (each coefficient in the orthonormal expansion is either $+\sqrt{E}$ or $-\sqrt{E}$. Assume as well that the error probability is related to the block length, energy per dimension and code rate via the cutoff rate theorem (soft decisions). Find (and plot) the throughput for code rates varying from 0.1 to 0.9 in steps of 0.1 as a function of the energy per information bit. (Use Matlab to plot the throughput). Assume $N = 500$. Be sure to normalize the energy per coded bit to the energy per information bit. Compare the throughput of hard decision decoding (BPSK and AWGN) and soft decision decoding.