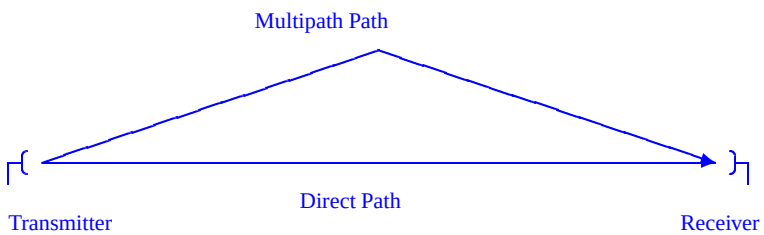
Lecture Notes 10: Fading Channels Models

In this lecture we examine models of fading channels and the performance of coding and modulation for fading channels. Fading occurs due to multiple paths between the transmitter and receiver. For example, two paths between the transmitter and receiver with approximately the same delay and amplitude but opposite phase (180 degree phase shift) will cause the signal amplitude to be essentially zero.



Fading Channel Models

The delay being approximately the same for the direct path and the multipath is relative to the duration of the transmitted symbol. That is the delay should be much shorter than the duration of the transmitted symbol. It may or may not be shorter than a single cycle of the carrier. Even if it is shorter than a single cycle the multipath could be due to a reflection from a building, the ground or trees and that can cause a phase reversal.

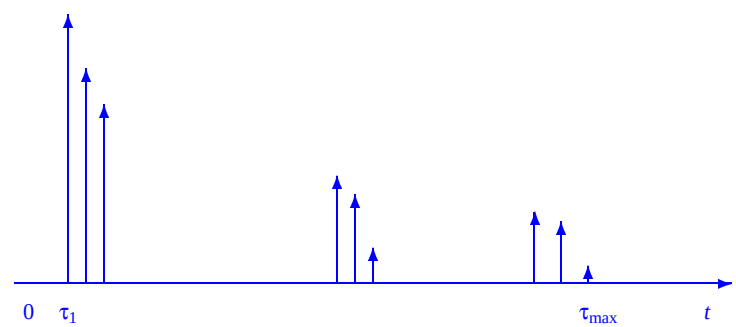
Fading Channel Models (cont.)

Thus the type of fading depends on various parameters of the channel and the transmitted signal. Fading can be considered as a (possibly) time varying filtering operation.

1. Frequency Selective Fading: If the transfer function of the filter has significant variations within the frequency band of the transmitted signal then the fading is called frequency selective.
2. Time Selective Fading: If the fading changes relatively quickly (compared to the duration of a data bit) then the fading is said to be time selective.
3. Doubly Selective: If both are true then it is said to be doubly selective.

Fading Channel Models (cont.)

Consider a simple model for fading where there are a finite number of paths from the transmitter to the receiver.



Consider a signal $u(t)$ transmitted over a channel with multipath fading. The signal will be represented as a baseband signal modulated onto a carrier

$$s(t) = \mathrm{Re}[s_0(t)\exp\{j2\pi f_c t\}]$$

where f_c is the carrier frequency (in radians) and $u_0(t)$ is the baseband signal. Alternatively the signal can be expressed as

$$s(t) = s_c(t)\cos(2\pi f_c t) - s_s(t)\sin(2\pi f_c t)$$

where

$$\begin{aligned} s_c(t) &= \mathrm{Re}[s_0(t)] \\ s_s(t) &= \mathrm{Im}[s_0(t)] \end{aligned}$$

Alternatively

$$s_0(t) = s_c(t) + js_s(t)$$

The signals $s_0(t)$, $s_c(t)$, $s_s(t)$ are assumed to have frequency response much lower than the carrier frequency.

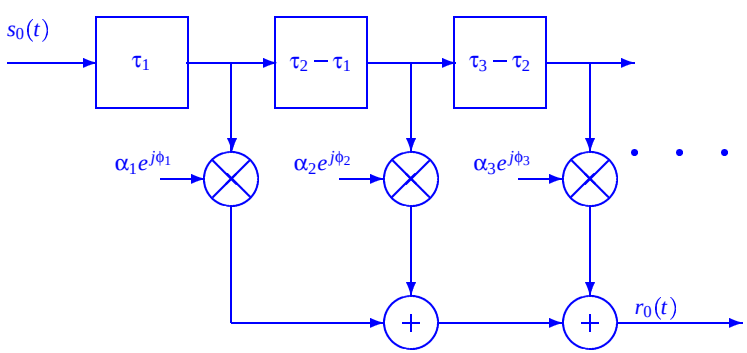
Fading Model

$$\begin{aligned} r(t) &= \sum_k \alpha_k s(t-\tau_k) \\ r(t) &= \mathrm{Re}[\sum_k \alpha_k s_0(t-\tau_k)\exp\{j2\pi f_c(t-\tau_k) + j\phi_k\}] \\ r(t) &= \mathrm{Re}[r_0(t)\exp\{j2\pi f_c t\}] \end{aligned}$$

where

$$r_0(t) = \sum_k \alpha_k s_0(t-\tau_k)\exp\{j\phi_k - j2\pi f_c \tau_k\}$$

Model for a Fading Channel



Example 1: Nonselective Fading

The multipath components are assumed to have independent phases. If we let W denote the bandwidth of the transmitted signal then the envelope of the signal does not change significantly in time smaller than $1/W$. Thus if $\tau_{\max} \ll 1/W$, that is

$$\frac{1}{f_c} \ll \tau_k \ll T = W^{-1}$$

then $s_0(t-\tau_k) \approx s_0(t)$. Then

$$r_0(t) = s_0(t)\left(\sum_k \alpha_k \exp\{j\theta_k\}\right)$$

where $\theta_k = \phi_k - 2\pi f_c \tau_k$. The factor $\sum_k \alpha_k \exp\{j\theta_k\}$ by which the signal is attenuated is (for a large number of paths with θ uniformly distributed) a Rayleigh distributed random variable. The fading occurs because of the random phases sometimes adding destructively and sometimes adding constructively. Thus for narrow enough bandwidths the multipath results in an attenuation by a Rayleigh distributed random variable.

This is likely to be the case for frequency-hopped spread-spectrum since the bandwidth is that of each hop, not the spread bandwidth.

Usually the paths lengths are changing with time due to motion of the transmitter or receiver. Above we have assumed that the motion is slow enough relative to the symbol duration so that the $\alpha_k(t)$ and $\phi_k(t)$ are constants.

Example 2: Frequency Selective Fading

If the bandwidth of the low pass signal is W or the symbol duration is $T = 1/W$ and the delays satisfy

$$\tau_k \gg T = W^{-1}$$

then the channel exhibits frequency selective fading. For example consider a discrete multipath model. That is,

$$r_0(t) = \alpha_1 e^{j\theta_1} s_0(t-\tau_1) + \cdots \alpha_M s_0(t-\tau_M) e^{j\theta_M}$$

The impulse response of this channel is

$$h(t) = \sum_{k=1}^{M} \alpha_k e^{j\theta_k} \delta(t-\tau_k)$$

The transfer function is

$$H(f) = \sum_{k=1}^{M} \alpha_k \exp\{j\theta_k - j2\pi f \tau_k\}$$

More specifically assume $M = 2$ and that the receiver is synchronized to the first path (so that

we assume $\tau_1 = \phi_1 = \theta_1 = 0$). Then

$$H(f) = 1 + \alpha_2 \exp\{j\theta_2 - j2\pi f \tau_2\}$$

Now it is clear that at frequencies where $2\pi f \tau_2 = \theta_2 + 2n\pi$ or $f = (\theta_2 + 2n\pi)/2\pi\tau_2$ the transfer function will be $H(f) = 1 + \alpha_2$ while at frequencies where $2\pi f \tau_2 = \theta_2 + (2n+1)\pi$ or $f = (\theta_2 + (2n+1)\pi)/2\pi\tau_2$ the transfer will be $H(f) = 1 - \alpha_2$.

The frequency range between successive nulls is $1/\tau$. Thus if $\tau \gg \frac{1}{W}$, $\frac{1}{\tau} \ll W$ implies there will be multiple nulls in the spectrum of the received signal.

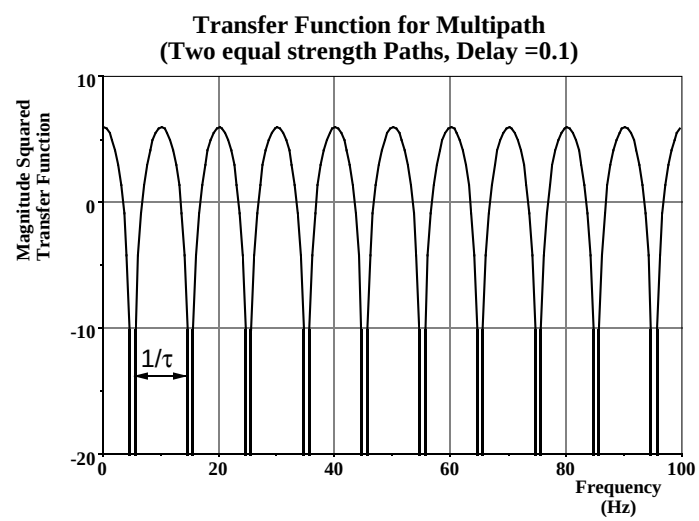


Figure 106: Transfer Function of Multipath Channel.

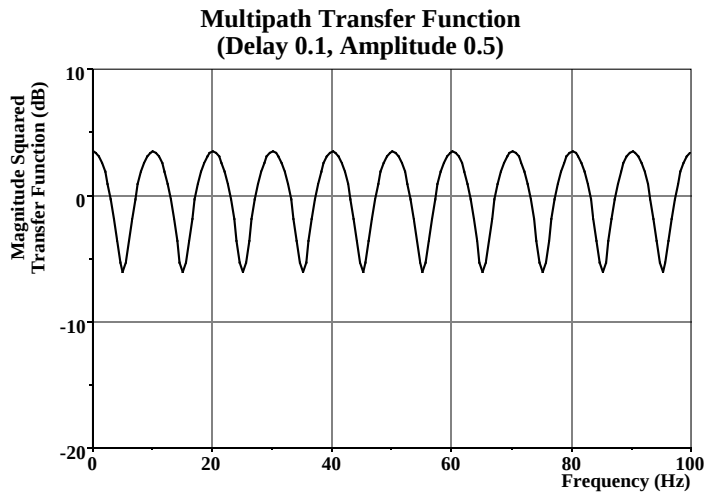


Figure 107: Transfer Function of Multipath Channel.

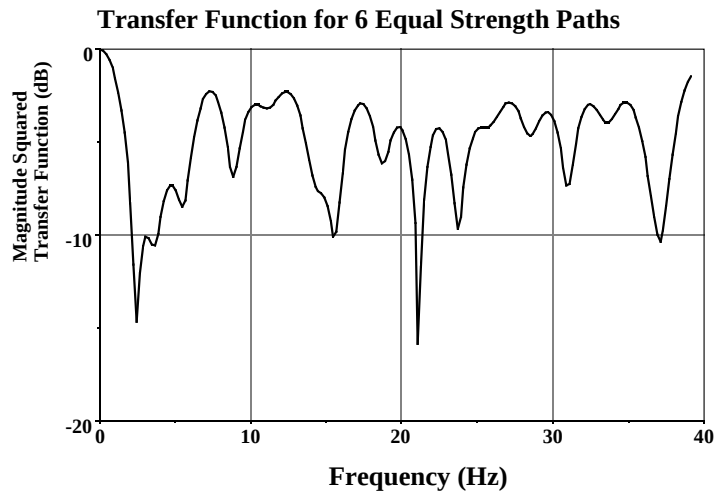


Figure 108: Transfer Function of Multipath Channel.

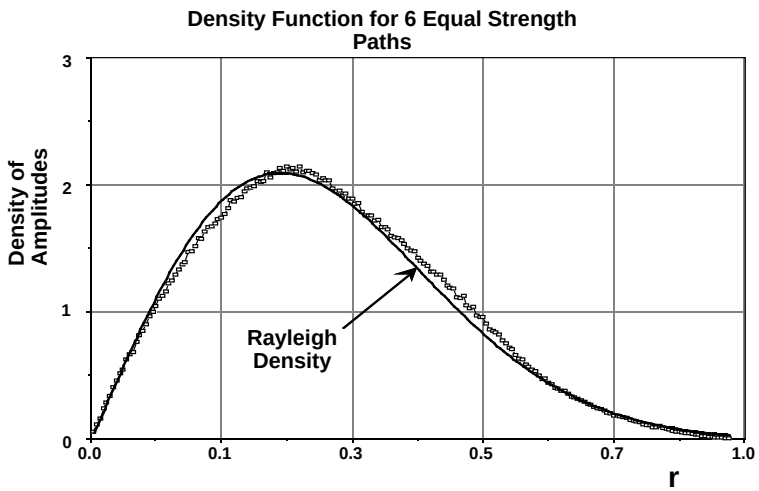


Figure 109: Density Function of Received Signal for a Channel with 6 Equal Amplitude Paths.

Example 3: Time Selective Fading

Now consider a communication system where the path length is changing as a function of time (e.g. due to vehicle motion). The envelope of the received signal (as the vehicle moves) undergoes time-dependent fading.

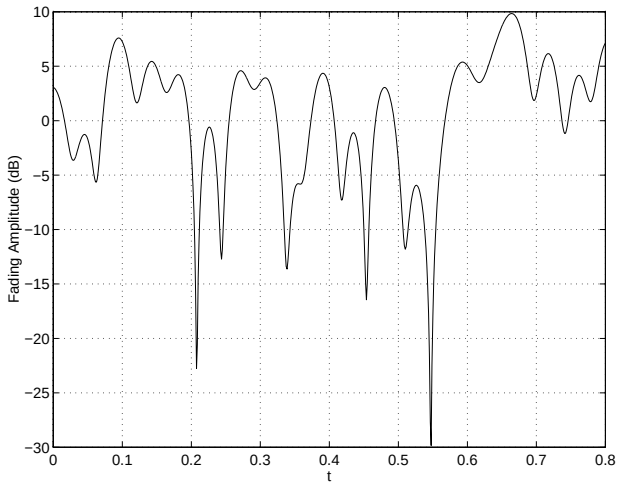


Figure 110: Received Signal Strength as a function of time for vehicle velocity 10mph.

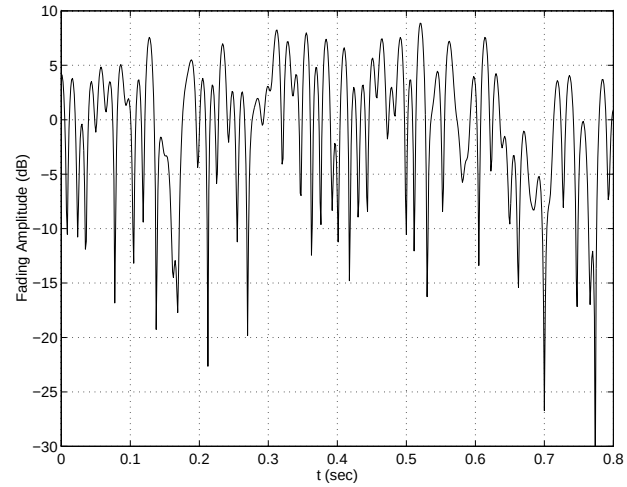


Figure 111: Received Signal Strength as a function of time for vehicle velocity 30mph.

Time Selective Fading Model

$$h(t) = \sum_{k=1}^{M} \alpha_k(t) e^{j\theta(t)\delta(t-\tau_k(t))}$$

Because the impulse response is time varying the fading at different time instances is correlated if they are very close and uncorrelated if they are very far apart. The time-varying response leads to a shift in the frequency spectrum. The spread in the spectrum of the transmitted signal is known as the Doppler spread. If the data duration is much shorter than the time variation of the fading process then the fading can be considered a constant or a slowly changing random process.

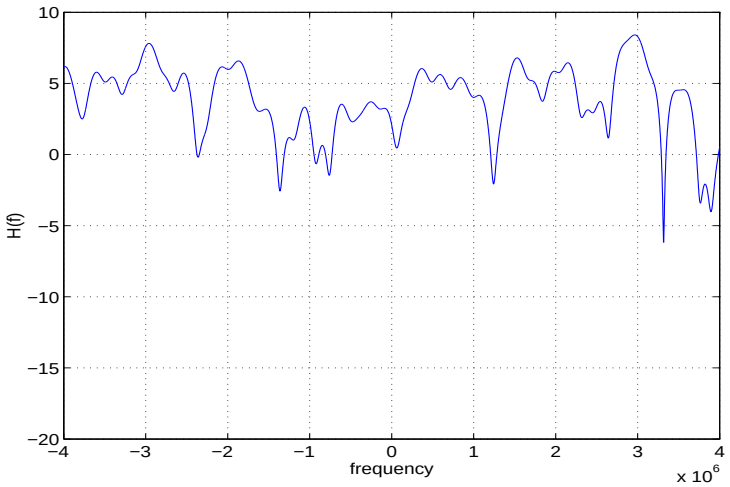
Fading Channels Models and Performance

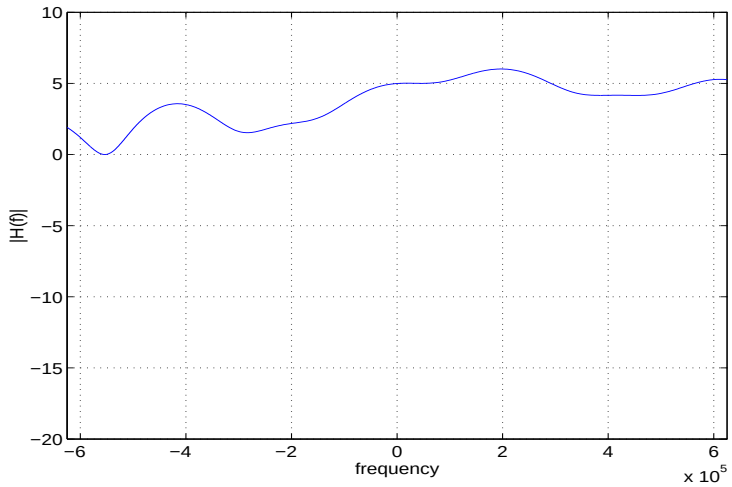
General Model

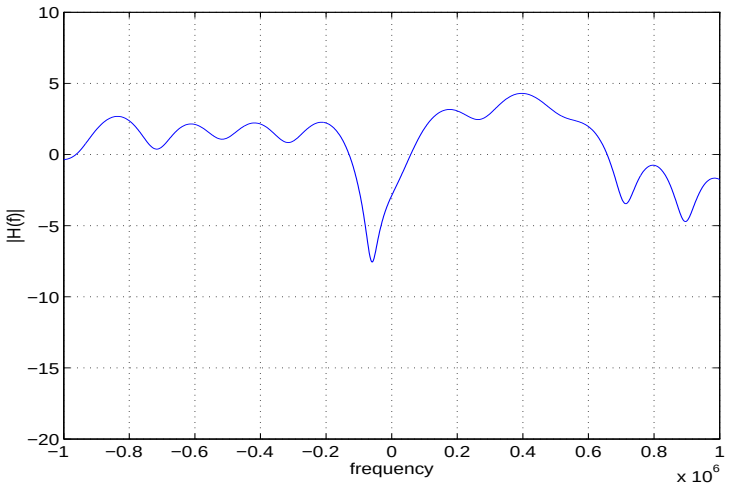
The most widely used general model for fading channels is the wide-sense stationary, uncorrelated scattering (WSSUS) fading model. In this model the received signal is modeled as a time-varying filter operation on the transmitted signal. That is

$$r_0(t) = \int_{-\infty}^{\infty} h(t;t-\alpha) s_0(\alpha) d\alpha$$

where $h(t;t-\tau)$ is the response due to an impulse at time τ and is modeled as a zero mean complex Gaussian random process. Note that it depends not only on the time difference between the output and the input but also on the time directly. The first variable in h accounts for the time varying nature of the channel while the second variable accounts for the delay between the input and output.







The response at a particular delay (and time) being Gaussian is a consequence of the assumption that there are a large number of (possibly time varying) paths at a *given* delay with independent phases which contibute to the overall response at that delay. When the channel response is a complex Gaussian random variable then the envelope of the response has a Rayleigh distribution. This is known as a Rayleigh faded channel.

If there is a (strong) direct path between the transmitter and receiver then the filter $h(t,\tau)$ will have nonzero mean. This case is called a Rician faded channel.

The assumptions for WSSUS is that $h(t,\tau)$

$$E[h(t;\tau_1)h^*(t+\Delta t;\tau_2)] = \phi(\tau_1;\Delta t)\delta(\tau_2-\tau_1).$$

Thus the correlation between the responses at two different times depends only on the difference between times (this is the wide sense stationary assumptions). Also, the response at two different delays are uncorrelated.

The amount of power received at a given delay τ is $\phi(\tau;0)$. This is called the intensity delay profile or the delay power spectrum.

The mean excess delay, μ_m is defined to be the average excess delay above delay of the first path

$$\mu = \frac{\int_{\tau_{min}}^{\tau_{max}} \tau\phi(\tau;0)d\tau}{\int_{\tau_{min}}^{\tau_{max}} \phi(\tau;0)d\tau} - \tau_{min}$$

The rms delay spread is defined as

$$s = [\frac{\int_{\tau_{min}}^{\tau_{max}} (\tau-\mu-\tau_{min})^2\phi(\tau;0)d\tau}{\int_{\tau_{min}}^{\tau_{max}} \phi(\tau;0)d\tau}]^{1/2}$$

The largest value $\tau_{\max}$ of τ such that $\phi(\tau;0)$ is nonzero is called the multipath spread of the channel. In the general model the delays cause distortion in the received signal. If we let

$H(f;t)$ be the time-varying transfer function of the channel, i.e.

$$H(f;t) = \int_{-\infty}^{\infty} h(t;\tau)e^{-j2\pi f\tau}d\tau$$

Then $H(f;t)$ is also a complex Gaussian random process. The correlation between the transfer function at two different frequencies is

$$\begin{aligned}\Phi(f_1,f_2;\Delta t) &= E[H(f_1;t)H^*(f_2;t+\Delta t)] \\ &= \int_{-\infty}^{\infty} \phi(\tau;\Delta t)e^{-j2\pi(f_2-f_1)\tau}d\tau\end{aligned}$$

Thus the correlation between two frequencies for the WSSUS model (and at two times) depends only on the frequency difference. If we let $\Delta t = 0$ then we obtain

$$\Phi(\Delta f;0) = \int_{-\infty}^{\infty} \phi(\tau;0)e^{-j2\pi(\Delta f)\tau}d\tau$$

The smallest frequency B_c separation such that the channel response is uncorrelated at that frequency difference is called the coherence bandwidth. It is related to the delay spread by

$$B_c = \frac{1}{\tau_{\max}}$$

Now consider the time-varying nature of the channel. In particular consider $\Phi(\Delta f;\Delta t)$ which is the correlation between the responses of the channel at two frequencies separated by Δf

and at times separated by Δt. For $\Delta f = 0$ $\Phi(0;\Delta t)$ measures the correlation between two responses (at the same frequency) but separated in time by Δt. The Fourier transform gives the Doppler power spectral density

$$S(\lambda) = \int_{-\infty}^{\infty} \Phi(0;\gamma)e^{-j2\pi\lambda\gamma}d\gamma$$

Consider a situation where a mobile is moving toward a base station with velocity v. If we assume that there are many multipath components that arrive with an angle uniformly distributed over $[0,2\pi]$ then the Doppler spectral density is given by

Example 1:

$$S(\lambda) = \frac{1}{\pi f_m}[1-(\lambda/f_m)^2]^{-1/2}, \quad 0 \le |\lambda| \le f_m$$

where $f_m = vf_c/c$, f_c is the center frequency and c is the speed of light ($3\times10^8 m/s$). For example a vehicle moving at 100mph with 1GHz center frequency has maximum Doppler shift of 57 Hz. A vehicle moving at 30mph would have a maximum Doppler shift of 17Hz. The corresponding autocorrelation function is the inverse Fourier transform and is given by

$$\begin{aligned}\Phi(0,\gamma) &= \int_{-\infty}^{\infty} S(\lambda)e^{j2\pi\lambda\gamma}d\lambda \\ &= J_0(2\pi f_m\gamma)\end{aligned}$$

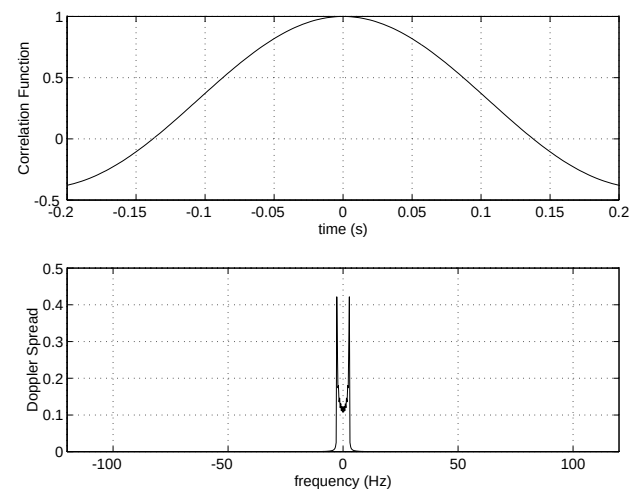


Figure 112: Channel Correlation Function and Doppler Spread for $f_c = 1$GHz, $v = 10$km/hour.

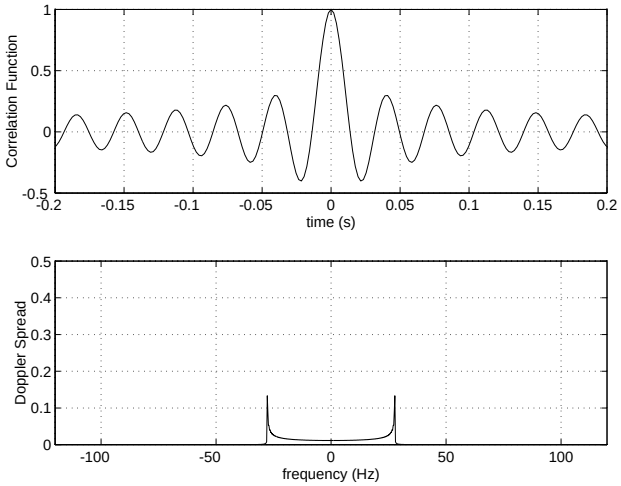
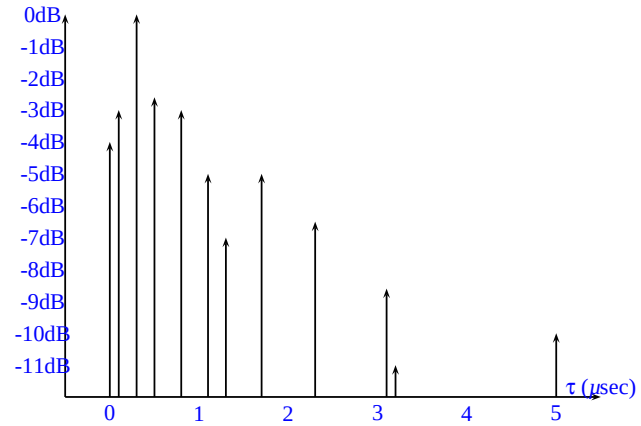


Figure 113: Channel Correlation Function and Doppler Spread for $f_c = 1$GHz, $v = 100$km/hour.

Example 2: If the channel is not time varying (i.e. time invariant) then the response at two different times are perfectly correlated so that $\Phi(0;\Delta t) = 1$ This implies that $S(\lambda) = \delta(f)$.

Example 3:



Doppler Spread, Coherence Time

The largest value of λ for which $S(\lambda)$ is nonzero is called the Doppler Spread of the channel. It is related to the coherence time T_c, the largest time difference for which the responses are correlated by

$$B_d = \frac{1}{T_c}$$

The fading discussed above is referred to as short term fading as opposed to long term fading. Long term fading refers to shadowing of the receiver from the transmitter due to terrain and buildings. The time scale for long term fading is much longer (on the order of seconds or minutes) than the time scale for short term fading. It is generally modeled as lognormal. That is the received power (in dB) has a normal (or Gaussian) distribution.

Free Space Propagation

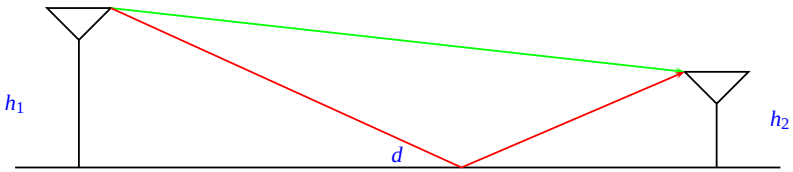
In this section we discuss the received power as a function of distance from the receiver. Suppose we have a transmitter and receiver separated by a distance d. The transmitter and receiver have antennas with gain G_t and G_r respectively. If the transmitted power is P_t the received power is

$$P_r = P_t G_r G_t \left(\frac{\lambda}{4\pi d}\right)^2$$

where $\lambda = c/f$ is the wavelength of the signal. The above equation holds in free space without any reflections or multipath of any sort.

Ground Reflection

Now consider the case where there is an additional path due to a signal reflection from the ground. The multipath has a different phase from the direct path. If we assume the reflection from the ground causes a 180 degree phase change then for large distances the relation between the transmitted power and the received power changes to



$$P_r \approx P_t G_r G_t \frac{h_1^2 h_2^2}{d^4} \quad (d \gg \max(h_1, h_2))$$

Thus the relation of received power to distance becomes an inverse fourth power law or equivalently the power decreases 40dB per decade of distance. Generally, for a wireless channel the decrease in power is 20dB per decade near the base station but as the receiver moves away the rate of decrease increases.

Summary of Models

Thus overall there are three degradations due to propagation.

- Path Loss. The first is the normal path loss (in free space this corresponds to received power decreasing as $1/r^2$ while in an urban environment this could be as $1/r^4$ where r is the distance between the transmitter and receiver).
- Long term shadowing. This has log-normal distribution (normal distribution for power received in dB). This is generally slowly varying (many symbols or even packets).
- Fast Fading. This is due to multipath components adding constructively or destructively.

BPSK Performance in Nonselective Fading

First consider a modulator transmitting a BPSK signal and received with a faded amplitude. The transmitted signal is

$$s(t) = \sqrt{2P} b(t) \cos(2\pi f_c t)$$

where $b(t)$ is the usual data bit signal consisting of a sequence of rectangular pulses of amplitude +1 or -1. The received signal is

$$r(t) = R\sqrt{2P} b(t) \cos(2\pi f_c t + \phi) + n(t)$$

Assuming the receiver can accurately estimate the phase the demodulator (matched filter) output at time kT is

$$z_k = R\sqrt{E} b_{k-1} + \eta_k$$

where $E = PT$. The random variable R represents the fading and has density

$$p_R(r) = \begin{cases} 0, & r < 0 \\ \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} & r \geq 0 \end{cases}$$

The conditional error probability is

$$P_e(R) = Q\left(\sqrt{\frac{2ER^2}{N_0}}\right).$$

Let $\alpha = 2E/N_0$, $\beta = \sigma^2\alpha = \bar{E}/N_0$ and

$$\begin{aligned} \gamma &= \sqrt{\frac{\beta}{1+\beta}} \\ &= \sqrt{\frac{\bar{E}/N_0}{1+E/N_0}} \end{aligned}$$

The unconditional error probability is

$$\begin{aligned} P_e &= \int_{r=0}^{\infty} p_R(r) Q\left(\sqrt{\frac{2Er^2}{N_0}}\right) dr \\ &= \int_{r=0}^{\infty} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} Q\left(\sqrt{\frac{2Er^2}{N_0}}\right) dr \\ &= \int_{r=0}^{\infty} \frac{r}{\sigma^2} \int_{u=\sqrt{\alpha r^2}}^{\infty} e^{-r^2/2\sigma^2} \frac{\exp\{-u^2/2\}}{\sqrt{2\pi}} du dr \end{aligned}$$

$$
\begin{aligned}
&= \int_{u=0}^{\infty} \frac{\exp\{-u^2/2\}}{\sqrt{2\pi}} \int_{r=0}^{u/\sqrt{\alpha}} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} du dr \\
&= \int_{u=0}^{\infty} \frac{\exp\{-u^2/2\}}{\sqrt{2\pi}} \int_{r=0}^{u/\sqrt{\alpha}} \frac{r}{\sigma^2} \exp\{-r^2/2\sigma^2\} dr du \\
&= \int_{u=0}^{\infty} \frac{\exp\{-u^2/2\}}{\sqrt{2\pi}} (1-\exp\{-u^2/(2\alpha\sigma^2)\}) du \\
&= \frac{1}{2} - \int_{u=0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{u^2}{2}(1+\frac{1}{\alpha\sigma^2})\} du \\
&= \frac{1}{2} - \gamma \int_{u=0}^{\infty} \frac{1}{\gamma\sqrt{2\pi}} \exp\{-\frac{u^2}{2\gamma^2}\} du \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{E}/N_0}{1+E/N_0}}
\end{aligned}
$$

The last integral is evaluated by recognizing the integrand to be a Gaussian density function with zero mean which when integrated from 0 to ∞ is 1/2.

For large E/N_0 the error probability is

$$P_e = \frac{1}{4E/N_0}.$$

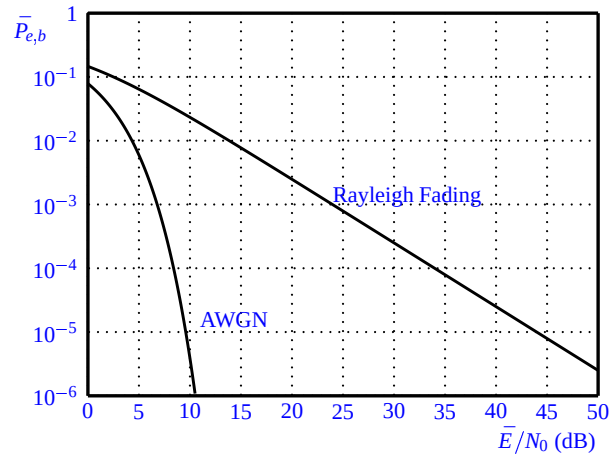


Figure 114: Bit error probability for BPSK with Rayleigh Fading

Performance

Thus for high E/N_0 the error probability decreases inverse linearly with signal-to-noise ratio. To achieve error probability of 10^{-5} require a signal-to-noise ratio of 43.0dB whereas in additive white Gaussian noise the required signal-to-noise ratio for the same error probability is 9.6dB. Thus fading causes a loss in signal-to-noise ratio of 33.4dB. This loss in performance is at the same average received power. The cause of this loss is the fact that the signal amplitude sometimes is very small and causes the error probability to be close to 1/2. Of course, sometimes the signal amplitude is large and results in very small error probability (say 0). However when we average the error probability the result is going to be much larger than the error probability at the average signal-to-noise ratio because of the highly nonlinear nature of the error probability as a function of signal amplitude without fading.

BPSK with Diversity

To overcome this loss in performance (without just increasing power) a number of techniques are applied. Many of the techniques attempt to receive the same information with independent fading statistics. This is generally called diversity. The diversity could be the form of L different antennas suitably separated so that the fading on different paths from the transmitter are independent. The diversity could be the form of transmitting the same data L times suitably separated in time so that the fading is independent.

In any case consider a system with L independent paths. The receiver demodulates each path coherently. Assume that the receiver also knows exactly the faded amplitude on each path. The decision statistics are then given by

$$z_l = r_l\sqrt{E}b + \eta_l, \quad l = 1,2,...,L$$

where r_l are Rayleigh, η_l is Gaussian and b represents the data bit transmitted which is either +1 or -1. The optimal method to combine the demodulator outputs can be derived as follows. Let $p_1(z_1,...,z_L|r_1,...,r_L)$ be the conditional density function of $z_1,...,z_L$ given the transmitted bit is +1 and the fading amplitude is $r_1,...,r_L$. The unconditional density is

$$p_1(z_1,...,z_L,r_1,...,r_L) = p_1(z_1,...,z_L|r_1,...,r_L)p(r_1,...,r_L)$$

The conditional density of z_1 given $b = 1$ and r_1 is Gaussian with mean $r_l\sqrt{E}$ and variance $N_0/2$. The joint distribution of $z_1, ..., z_L$ is the product of the marginal density functions. The optimal combining rule is derived from the ratio

$$\begin{aligned}
\Lambda &= \frac{p_1(z_1, ..., z_L, r_1, ..., r_L)}{p_{-1}(z_1, ..., z_L, r_1, ..., r_L)} \\
&= \frac{p_1(z_1, ..., z_L | r_1, ..., r_L) p(r_1, ..., r_L)}{p_{-1}(z_1, ..., z_L | r_1, ..., r_L) p(r_1, ..., r_L)} \\
&= \frac{p_1(z_1, ..., z_L | r_1, ..., r_L)}{p_{-1}(z_1, ..., z_L | r_1, ..., r_L)} \\
&= \frac{\exp\{-\frac{1}{N_0}\sum_{l=1}^{L}(z_l - r_l\sqrt{E})^2)}{\exp\{-\frac{1}{N_0}\sum_{l=1}^{L}(z_l + r_l\sqrt{E})^2)} \\
&= \exp\{\frac{4}{N_0}\sum_{l=1}^{L} z_l r_l \sqrt{E}\}.
\end{aligned}$$

The optimum decision rule is to compare Λ with 1 to make a decision. Thus the optimal rule is

$$\sum_{l=1}^{L} r_l z_l \underset{b=-1}{\overset{b=+1}{\gtrless}} 0$$

The error probability with diversity L can be determined using the same technique as used

without diversity. The expression for error probability is

$$P_e(L) = P_e(1) - \frac{1}{2}\sum_{k=1}^{L-1}\frac{(2k)!}{k!k!}(1 - 2P_e(1))(P_e(1))^k(1 - P_e(1))^k$$

In the case of diversity transmission the energy transmitted per bit E_b is LE. For a fixed E_b as L increases each transmission contains less and less energy but there are more transmissions over independent faded paths. In the limit as L becomes large using the weak law of large numbers it can be shown that

$$\lim_{L\to\infty} P_e(L) = Q\left(\sqrt{\frac{2\bar{E}_b}{N_0}}\right)$$

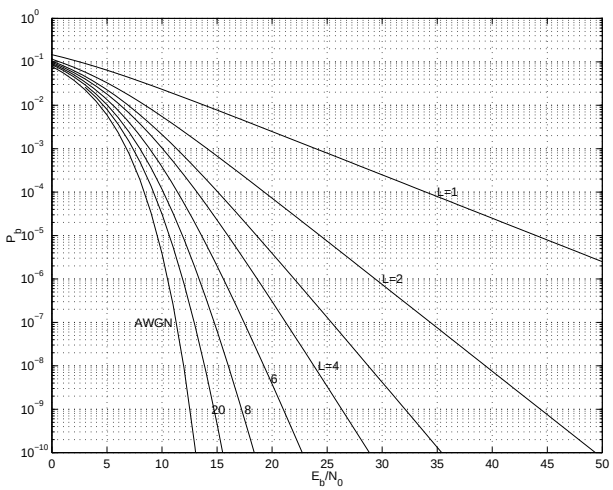


Figure 115: Error Probability for BPSK (coherent demodulation) with and without Rayleigh Fading.

Fading and Coding

In Figure 116 we show the performance of a rate 1/2 constraintl length 7 convolutional code on a Rayleigh faded channel (independent fading on each bit) where the receiver knows the fading level for each bit and can appropriatel weight the metric in the decoder. Notice that the requied E_b/N_0 for 10^{-5} bit error probability is about 7.5dB, which is less than that required for uncoded BPSK without fading. The gain compared to uncoded performance is more than 36dB.

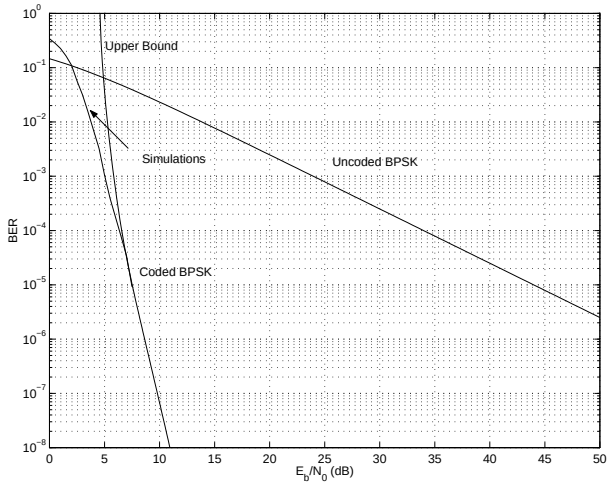


Figure 116: Error probability for BPSK (coherent demodulation) with Rayleigh fading and convolutional codeing.

Fundamental Limits

The fundamental limits on performance can be determined for a variety of circumstances. Here we assume that the transmitter has no knowledge of the fading amplitude and assume the modulation in binary phase shift keying. When the receiver knows exactly the amplitude (and phase) of the fading process the maximum rate of transmission (in bits/symbol) is

$$C = 1 - \int_{r=0}^{\infty} \int_{y=-\infty}^{\infty} f(r) g(y) \log_2(1 + e^{-2y\beta}) dy dr$$

where $f(r) = 2r \exp\{-r^2\}$, $\beta = \sqrt{2E/N_0}$ and

$$g(y) = \frac{1}{\sqrt{2\pi}} \exp\{-(y - \sqrt{2\bar{E}r^2/N_0})^2/2\}.$$

If the receiver does not know the fading amplitude (but still does coherent demodulation) then the capacity is

$$C = 1 - \int_{r=0}^{\infty} \int_{y=-\infty}^{\infty} p(y|1) \log_2(1 + \frac{p(y|0)}{p(y|1)}) dy$$

where

$$p(y|0) = \int_0^{\infty} f(r) \frac{1}{\sqrt{2\pi N_0}} e^{-(y-\sqrt{E}r)^2/N_0} dr$$

and

$$p(y|1) = \int_0^{\infty} f(r) \frac{1}{\sqrt{2\pi N_0}} e^{-(y+\sqrt{E}r)^2/N_0} dr$$

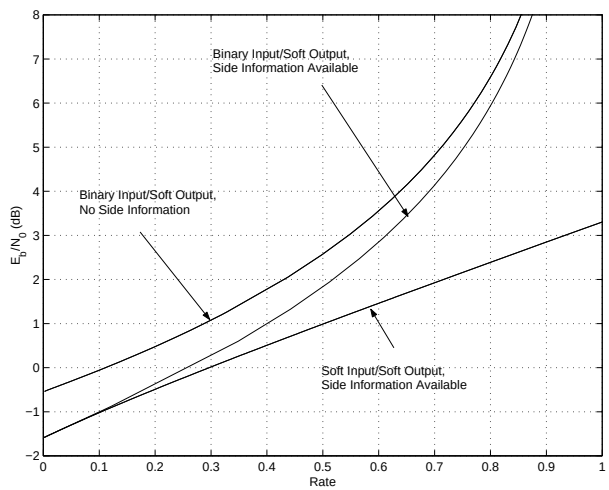


Figure 117: Capacity of Rayeigh Fading Channels.

If the receiver makes a hard decision and the receiver knows the fading amplitude then the

capacity is

$$C = \int_0^\infty f(r)\,[1 + p(r)\log_2(p(r)) + (1-p(r))\log_2(1-p(r))]\,dr$$

where $p(r) = Q(\sqrt{\frac{2\bar{E}r^2}{N_0}})$. For a receiver that does not know the fading amplitude and makes hard decisions on each coded bit the capacity is given by

$$C = 1 + \bar{p}\log_2(\bar{p}) + (1-\bar{p})\log_2(1-\bar{p})$$

where

$$\bar{p} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{E}/N_0}{E/N_0 + 1}}.$$

Finally if the transmitter is not restricted to binary phase shift keying but can use any type of modulation then the capacity when the receiver knows the fading level is

$$C = \int_0^\infty f(r)\frac{1}{2}\log_2(1 + 2\bar{E}r^2/N_0)dr$$

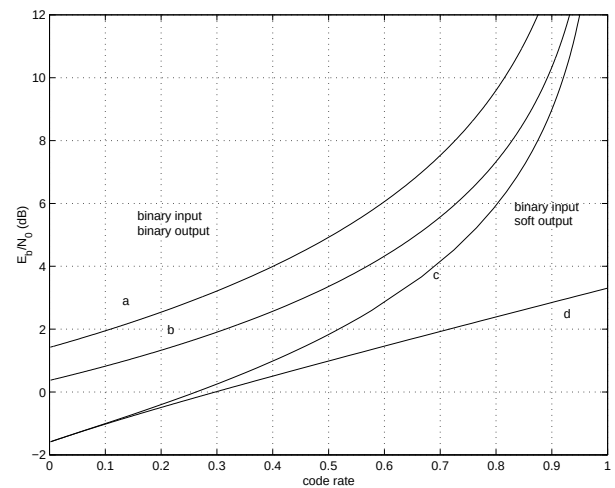


Figure 118: Capacity of Rayleigh faded channel with coherent detection

In this figure the top curve (a) is the minimum signal-to-noise ratio necessary for reliable

communication with hard decisions, no side information. The second curve (b) is the case of hard decisions with side information. The third curve (c) is the case of soft decisions with side information and binary modulation (BPSK). The bottom curve (d) is the case of unrestricted modulation and side information available a the reveiver. There is about a 2 dB gap between hard decision and soft decisions when side information is available. There is an extra one dB degradation in hard decisions if the receiver does not know the amplitude. A roughly similar degradation in performance is also true for soft decisions with and without side information.

We can also plot the maximum rate of communications in bits/second/Hz as a function of the energy per bit to noise power ratio.

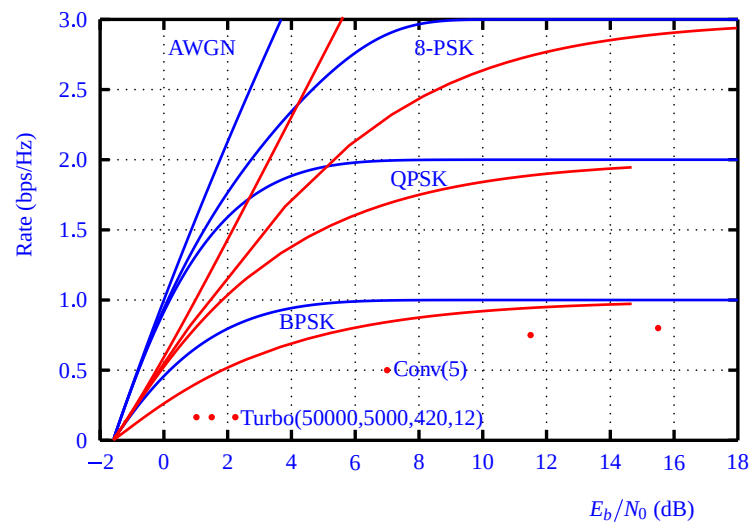


Figure 119: Transmission Rate versus Signal-to-Noise Ratio

Noncoherent Demodulation

In this section consider noncoherent demodulation of FSK. The transmitted signal is

$$s(t) = \sqrt{2P}\sum_{l=-\infty}^{\infty}\cos(2\pi(f_c+b(t)\Delta f)t+\theta)p_T(t-lT).$$

The received signal is now

$$r(t) = \sqrt{2P}R\cos(2\pi(f_c+b(t)\Delta f)t+\theta+\psi)+n(t)$$

where ψ is the additional phase due to the fading. and the fading level R is a Rayleigh distributed random variable. The density of R is

$$p_R(r) = \begin{cases} 0, & r<0 \\ \frac{r}{\sigma^2}e^{-r^2/2\sigma^2} & r\geq 0 \end{cases}$$

where $2\sigma^2$ is the mean square value of the fading variable. If we let $\bar{E}$ denote the received energy (as opposed to the transmitted energy E) then

$$\bar{E} = 2\sigma^2 E.$$

The outputs of the standard FSK receiver are

$$\begin{aligned} X_{c,1}(iT) &= \sqrt{E}R\delta(b_{i-1},1)\cos(\theta)+\eta_{c,1,i} \\ X_{s,1}(iT) &= \sqrt{E}R\delta(b_{i-1},1)\sin(\theta)+\eta_{s,1,i} \\ X_{c,-1}(iT) &= \sqrt{E}R\delta(b_{i-1},-1)\cos(\theta)+\eta_{c,-1,i} \\ X_{s,-1}(iT) &= \sqrt{E}R\delta(b_{i-1},-1)\sin(\theta)+\eta_{s,-1,i} \end{aligned}$$

The conditional error probability is then

$$P_e(R) = \frac{1}{2}e^{-RE_b/N_0}$$

The unconditional error probability is

$$\begin{aligned} P_e &= \int_0^\infty p_R(r)\frac{1}{2}e^{-rE_b/N_0}dr \\ &= \frac{1}{2+2\sigma^2E/N_0} = \frac{1}{2+E/N_0} \end{aligned}$$

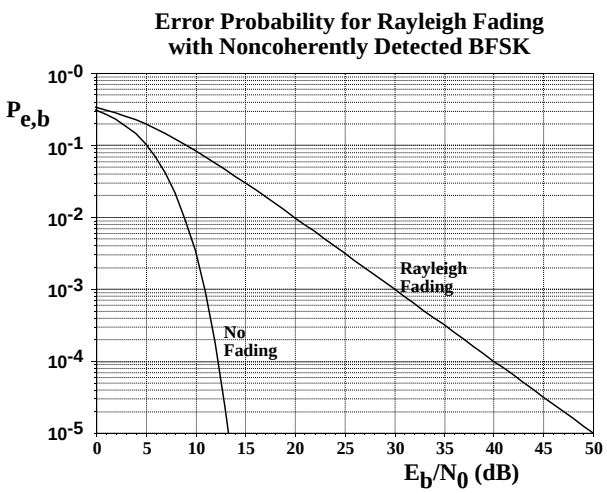


Figure 120: Error Probability for Noncoherent Detection of FSK with Rayleigh Fading.

There is an increase of 36.6dB in the signal-to-noise ratio needed for 10^{-5} error probability when Rayleigh fading is present.

M-ary Orthogonal Modulation

The above result is easily extended to M-ary orthogonal signalling. The symbol error probability is

$$P_e(M) = \sum_{m=1}^{M-1} \binom{M-1}{m} (-1)^{m+1} \frac{1}{1+m+mE/N_0}$$

For large M this is difficult to compute. Asymptotically the error probability is

$$\lim_{M\to\infty} P_e(M) = 1 - \exp\{-\ln(2)/\bar{E_b}/N_0\}$$

where $E_b = E/\log_2(M)$ is the energy transmitted per bit of information. This asymptotic result also holds with coherent detection.

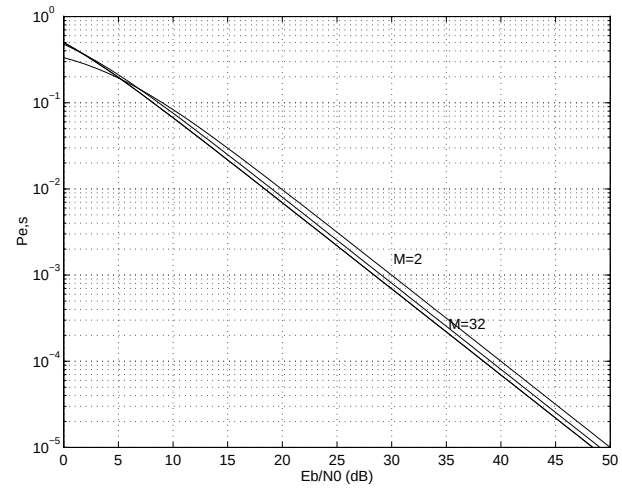


Figure 121: Symbol error probability for M-ary orthogonal signalling with Rayleigh fading ($M = 2, 32, \infty$).

A word of caution is necessary here. For different M the time bandwidth product of the modulation changes. If the time is kept constant then larger M implies larger bandwidth. The assumption that the channel is frequency nonselective over the larger bandwidth should be reconsidered. If the channel becomes frequency selective then not all of the signal will fade simultaneously (as in the assumption above). On the other hand, there will be some intersymbol interference from the frequency selectivity. Careful analysis of the affect of frequency selectivity will be discussed after considering a spread-spectrum waveform.

Rician Fading

When there is a direct path between the transmitter and receiver in addition to the many reflected paths then the channel is said to undergo Rician fading.

The transmitted signal for FSK it is expressed as

$$s(t) = \sqrt{2P}\cos(2\pi(f_c + b(t)\Delta f))$$

The received signal is now

$$r(t) = \sqrt{2P}\alpha\cos(2\pi(f_c + b(t)\Delta f) + \theta) + \sqrt{2P}R\cos(2\pi(f_c + b(t)\Delta f)t + \theta + \psi) + n(t)$$

where α represents the amplitude of the direct path component and R represents the fading amplitude for the multipath and ψ the additional phase due to the fading.

The outputs of the standard FSK receiver are

$$\begin{aligned}
X_{c,1}(iT) &= (\alpha + R)\sqrt{E}\delta(b_{i-1}, 1)\cos(\theta) + \eta_{c,1,i} \\
X_{s,1}(iT) &= (\alpha + R)\sqrt{E}\delta(b_{i-1}, 1)\sin(\theta) + \eta_{s,1,i} \\
X_{c,-1}(iT) &= (\alpha + R)\sqrt{E}\delta(b_{i-1}, -1)\cos(\theta) + \eta_{c,-1,i} \\
X_{s,-1}(iT) &= (\alpha + R)\sqrt{E}\delta(b_{i-1}, -1)\sin(\theta) + \eta_{s,-1,i}
\end{aligned}$$

The total received energy due to the transmitted signal is a random variable. The average received energy is

$$\bar{E} = (\alpha^2 + 2\sigma^2)E$$

An important quantity in quantifying performance is the ratio of the power in the faded part with the power in the direct path.

$$\gamma^2 = \frac{2\sigma^2}{\alpha^2}$$

The error probability for Rician fading can be expressed as follows.

$$P_{e,b} = \frac{1+\gamma^2}{2+\gamma^2(2+\Gamma)} \exp\{-\frac{\Gamma}{2+\gamma^2(2+\Gamma)}\}$$

where Γ is the ratio of the total received (direct plus faded) energy to the noise spectral density (N_0).

$$\Gamma = \frac{\bar{E}}{N_0}$$

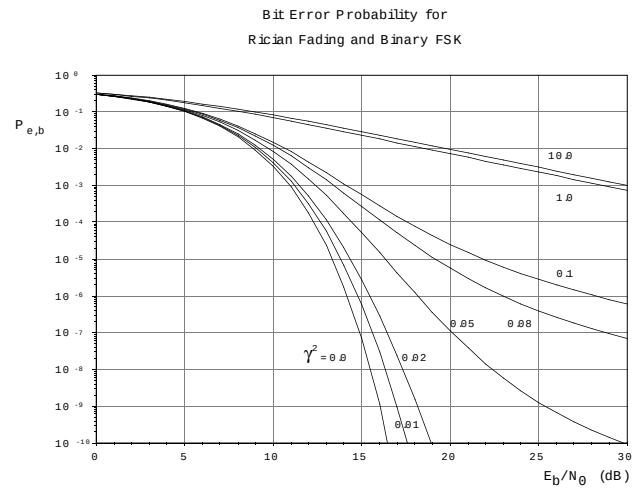


Figure 122: Error Probability for Binary FSK with Noncoherent Detection and Rician Fading.

Notice that if the faded power is equal or larger than the power of the direct path then the performance is essentially that of a pure Rayleigh fading channel. For 32-ary modulation the symbol error probability is shown in the following figure.

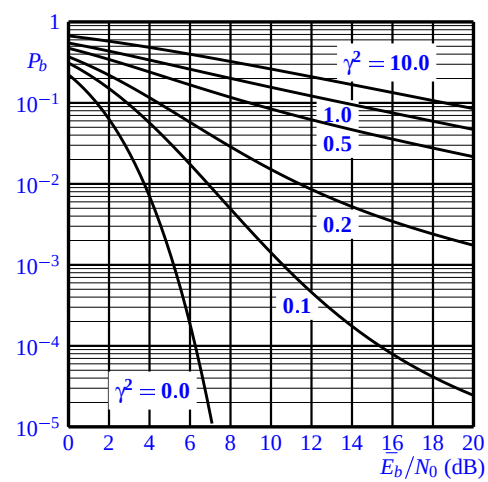


Figure 123: Symbol error probability of 32-ary orthogonal modulation in a Rician faded channel with noncoherent demodulation

BFSK with Diversity and Rayleigh Fading

Now consider the case where we have L independent paths between the transmitter and receiver, i.e diversity. Assume the receiver has no knowledge of the fading amplitude or phase. In this case the optimal combining rule is called square-law combining. The error probability can be determined as

$$P_e(L) = P_e(1)^L \sum_{j=0}^{L-1} \binom{L+j-1}{j} (1-P_e(1))^j \leq D^L = [4P_e(1)(1-P_e(1))]^L$$

where

$$P_e(1) = \frac{1}{2+E/N_0}$$

For the case of diversity transmission $E = E_b/L$. The larger L is the small amount of energy in each diversity transmission. However, the larger L is, the more transmissions there are over independent faded paths. For noncoherent demodulation, unlike coherent demodulation, as L approaches ∞ the error probability approaches 1/2. However, for a fixed signal-to-noise ratio $\bar{E}_b/N_0$, there is an optimal value of L that minimizes the error probability. The optimal L is an increasing function of $\bar{E}_b/N_0$. If we do not worry about the fact that L must be an integer and

set $L = a\bar{E}_b/N_0$ then the error probability can be upper bounded by an exponentially decreasing function of $\bar{E}_b/N_0$.

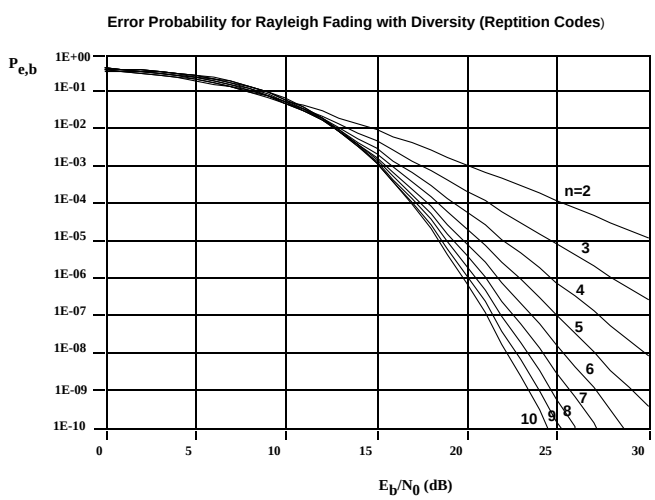


Figure 124: Error probability for repetition code (diversity) with Rayleigh fading.

It can be shown (using the Chernoff bound) that the error probability can be upper bound by an exponentially decreasing function of the signal-to-noise ratio.

$$P_{e,b} \leq e^{-0.149 E_b/N_0}.$$

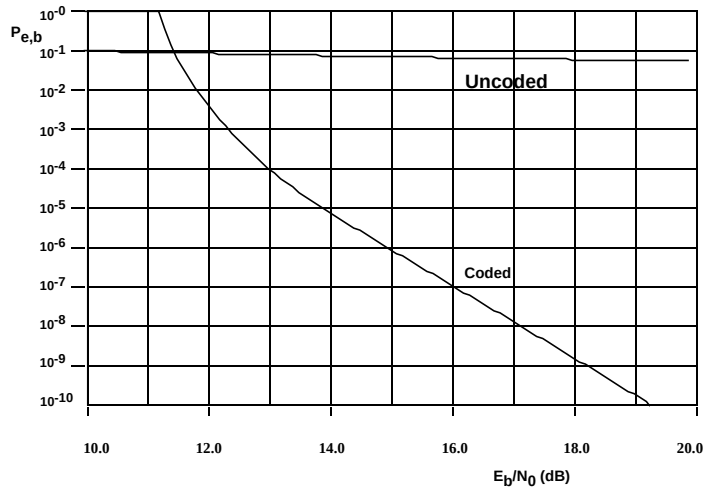


Figure 125: Error probability for rate 1/2 constraint length 7 convolutional code with Rayleigh fading (soft decision decoding).

As can be seen from the above figures it is extremely important that some form of coding be used with fading. When a repetition code is used the error probability can be made to decrease exponentially with signal to-noise ratio provided that we use the optimal diversity. Below we determine the capacity of the channel. We show that Rayleigh fading only causes a degradation of 1.35dB relative to unfaded system when optimal codes are applied (to both). A drawback to this is that the codes that achieve capacity at the minimum signal-to-noise ratio for Rayleigh fading have rate close to 0.24 while for an unfaded channel the rate is 0.48.

When we consider the output of the noncoherent matched filter to be the channel output and the input to be the binary symbol the capacity of the channel is given by

$$C = 1 - \frac{\exp\left\{-\frac{\alpha^2\beta^2/2}{1+\sigma^2\beta^2}\right\}}{4(1+\sigma^2\beta^2)} \int_0^\infty \int_0^\infty e^{-(y_1+y_{-1})/2} I_0[\sqrt{\alpha^2\beta^2 y_{-1}}] \log_2(1+\Lambda(y_{-1},y_1)) dy_{-1} dy_0$$

where $\beta^2 = \frac{2E}{N_0}$.

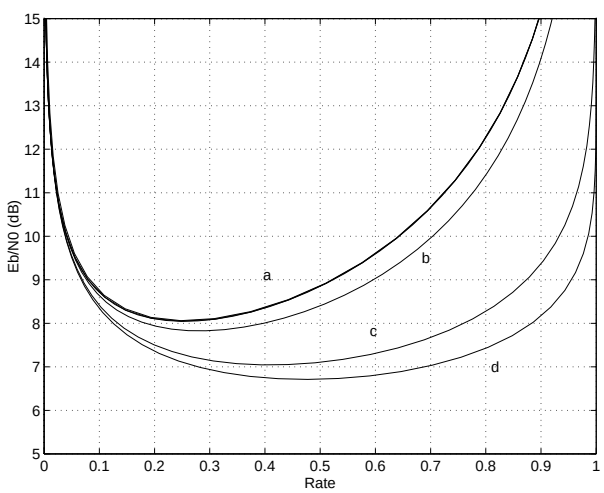


Figure 126: E_b/N_0 needed for reliable communication for Rician fading with soft decision decoding a) $\gamma^2 = \infty$, b) $\gamma^2 = 1.0$, c) $\gamma^2 = 0.1$, d) $\gamma^2 = 0.0$.

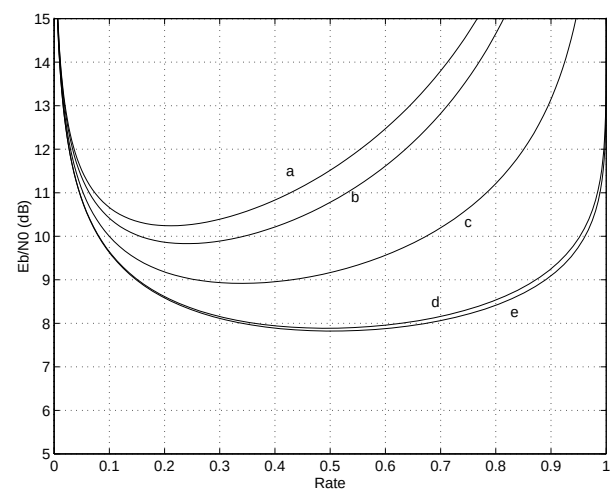


Figure 127: E_b/N_0 needed for reliable communication for Rician fading with hard decision decoding a) $\gamma^2 = \infty$, b) $\gamma^2 = 1.0$, c) $\gamma^2 = 0.5$, d) $\gamma^2 = 0.1$, e) $\gamma^2 = 0$.

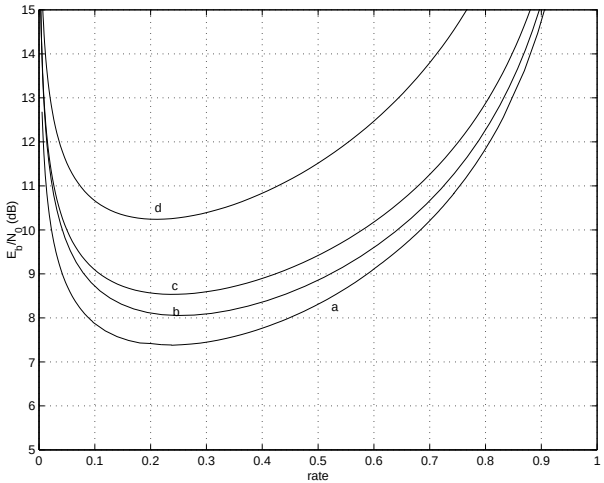


Figure 128: E_b/N_0 needed for reliable communication for Rayleigh fading (a) soft decisions with side information, (b) soft deicisoins without side information, (c) hard decisions with side information, (d) hard decisions without side information.