

# Capacity of Frequency-Hop Spread-Spectrum Multiple-Access Communication Systems

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## Abstract

The information theoretic capacity of frequency-hop spread-spectrum multiple-access communication is considered. In order to account for independent encoding and decoding and private (to the sender and receiver) hopping patterns an interference channel model is adopted with  $K$  sender-receiver pairs with the  $i$ -th receiver only interested in the message transmitted by the  $i$ -th sender. Both synchronous and asynchronous hopping patterns are investigated. Although the channel exhibits memory in the latter case, we are able to compute the capacity region. The asymptotic normalized sum capacity is also computed.

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## I. Introduction

In this paper we consider the multiple-access capability of frequency-hop spread-spectrum communication systems from an information theoretic viewpoint. This capability is calculated by modelling the communication system from the modulator input to the demodulator output as an interference channel and determining the capacity region of this channel. We examine synchronous and asynchronous hopping patterns and consider the cases of side information at the receiver and no side information at the receiver.

There have been several recent studies of multiple-access performance of spread-spectrum communication systems [1,3]. These have concerned themselves with the probability of error of packets or codewords over such multiple-access channels using specific codes, typically Reed-Solomon codes. For the (much harder to analyze) unslotted channel they have used bounding techniques to approximate the packet or codeword error probability [4]. In this paper we address the issue of performance with the best possible codes. Here we examine only the slotted channel with both synchronous and asynchronous hopping each with and without side information and determine the capacity region.

We do not allow cooperation between users either at the encoder or at the decoder. This makes the interference channel model more appropriate to our situation rather than the multiple-access channels. Using this model we are able to calculate the capacity region in each of the cases mentioned above. We then focus our attention on the largest possible total rate that can be achieved by all

users.

## II. Channel Models

The model for multiple-access frequency-hop spread-spectrum communication consists of  $K$  source-receiver pairs with the  $i$ -th source desiring to communicate only with its corresponding receiver over a common channel (See Fig. 1). There are  $K$  separate encoder devices: one for each source. The  $i$ -th encoder has as its input only the messages from the  $i$ -th source and produces a symbol  $x^{(i)} \in \mathbf{X}$  (the common input alphabet). This symbol is transmitted by modulating and frequency-hopping the desired symbol. The  $i$ -th receiver examines its assigned hopping pattern demodulates the received waveform and produces the output symbol  $y^{(i)} \in \mathbf{Y}$  (the common output alphabet). Decoding is done independently at the  $i$ -th receiver. The  $i$ -th source may transmit one of  $\lfloor 2^{nR_i} \rfloor$  messages and this is then encoded by the  $i$ -th encoder, modulated using one of  $M$  signals, and then frequency hopped by the  $i$ -th frequency hopper to one of  $q$  frequency slots. The hopping patterns are modelled by independent sequences equiprobable over the  $q$  slots. Thus each component of each of the  $K$  input vectors of length  $n$  is chosen from the common alphabet  $\{1, 2, \dots, M\}$ . We assume  $l$  channel symbols are transmitted per hop and this is incorporated into the channel alphabet size  $M$ . We assume that the channel is slotted and thus the number of transmissions during a slot is constant.

The hopping patterns we consider are modelled by independent sequences,

one for each sender-receiver pair, equiprobable over the  $q$  slots. In the case of synchronous hopping this makes the channel hits (i.e. the event of more than one user transmitting over the same frequency slot) independent. However, in the case of asynchronous hopping patterns, knowledge about past hits by a particular user affects this user's knowledge about the distribution of the frequency slots used by the  $K - 1$  other users and so the sequence of hits for any particular user exhibits memory. Moreover, it turns out that this sequence is not even Markovian [5]. In this paper we demonstrate that this sequence is a function of an underlying Markov chain which enables us to treat the marginal channels in the asynchronous case as finite state channels thereby allowing us to compute the capacity regions of the  $K$ -user channel with asynchronous hopping in both the case with side information at the receiver (knowledge about whether each received symbol was hit or not) and the case with no side information at the receiver. We consider only the noise arising from interference due to other users and do not include any other background noise in our analysis (although it would not be difficult to do so).

### **Case A: Synchronous Hopping-Side Information Available**

We first examine the case of synchronous hopping patterns and thus consider a memoryless channel model (since the symbol hits are independent in this case). The side information referred to is the awareness of each of the  $K$  decoders about whether or not there was a hit on the corresponding transmission. The symbols hit are erased. This model is shown in Fig. 2(a). It is easy to see that

$$\epsilon_A = P\{\text{erasure}\} = 1 - (1 - p)^{K-1} \quad (1)$$

where  $p = 1/q$ .

### Case B: Synchronous Hopping-No Side Information Available

Here again synchronous hopping is considered but in this case the decoders do not receive any information about hits on each symbol. Thus the hits remain undetected and cause a symbol error with probability  $e_0$  (see Fig. 2(b))

$$\epsilon_B/(M - 1) = P\{\text{error}\} = (1 - (1 - p)^{K-1})e_0/(M - 1). \quad (2)$$

Note that in our model we do not distinguish between the case of two users colliding or more than two users colliding.

### Case C: Asynchronous Hopping-Side Information Available

We now address the situation where the hopping is asynchronous and the receivers have side information which enables the demodulator and decoder to determine which symbols have been hit. It is assumed that all symbols that have been hit are erased. We need to introduce some notation which we do with the aid of Fig. 3. Each user employs a hopping pattern with frequencies chosen uniformly from the set  $\{1, \dots, q\}$  and independently of the frequencies chosen by the other users. We denote the random hopping pattern for user  $i$  as  $\{F_{i,j}, j = 0, 1, \dots\}$ . (All capital letters will denote random quantities (variables or vectors), and the corresponding lower-case letters will denote particular realizations of these random quantities). Observe that 2 channel symbols of user  $i$  overlap with the

$j$ -th channel symbol of user 1. We define the frequency possibly interfering with the transmission by user one in the  $j$ -th hop on the right (see Fig. 3) as  $F_{i,j}$  and  $(F_{2,j}, \dots, F_{K,j})$  as  $\tilde{F}_j$ . Now suppose all  $K$  users are transmitting packets and receiver 1 (which desires to receive user 1's messages) locks onto user 1's hopping pattern. We assume that user 1 transmits using frequencies  $F_{1,j}, F_{2,j}, \dots, F_{n,j}$ . We denote by  $H_j, j = 1, \dots, n$ ,  $n$  binary random variables such that  $H_j = 1$  if the  $j$ -th symbol transmitted by user 1 is hit (i.e. if at least one of the other  $K - 1$  users uses the same frequency during the slot corresponding to the  $j$ th channel symbol) and  $H_j = 0$  otherwise. We note that  $H_j = 1$  if and only if  $F_{1,j} \in \tilde{F}_{j-1} \cup \tilde{F}_j$ . We also will need the binary random variables  $H_j^L, H_j^R, j = 1, 2, \dots, n$  defined as follows

$$H_j^L = \begin{cases} 1 & F_{1,j} \in \tilde{F}_{j-1} \\ 0 & \text{otherwise,} \end{cases}$$

$$H_j^R = \begin{cases} 1 & F_{1,j} \in \tilde{F}_j \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let  $S_i^j$  be the number of frequency slots out of time slots  $i, i + 1, \dots, j$  of user 1 that have been hit.

Note that in the model we have just described the set  $\{F_{i,j}; i = 1, 2, \dots, K, j = 0, 1, \dots, n\}$  is an independent identically distributed (i.i.d.) set of random variables with each  $F_{i,j}$  being uniformly distributed on the set  $1, 2, \dots, q$  and consequently  $\{\tilde{F}_j; j = 0, 1, \dots, n\}$  is an i.i.d. sequence. Also  $H_j = 1$  if and only if  $H_j^L = 1$  or  $H_j^R = 1$  and  $S_i^j = H_i + \dots + H_j$ . Now we can state our key lemma.

Lemma 1: The sequence  $(H_j^L, H_j^R)$  is a Markov chain.

Proof: See Appendix A.

Note: In the sequence  $(\dots H_{j-1}^L, H_{j-1}^R, H_j^L, H_j^R, \dots)$ ,  $H_j^L$  depends on  $H_{j-1}^R$  but is independent of  $H_{j-1}^L$  and  $H_j^R$ . The reason why  $H_j$  is not a Markov chain (for  $K > 2$ ) is that knowledge of  $H_{j-1}, H_{j-2}, \dots$  affects our posteriori distribution of  $\tilde{F}_k, k = j - 1, j - 2, \dots$  and thereby  $H_{j-1}^R$  thus making

$$P(H_j | H_{j-1}, H_{j-2}, \dots) \neq P(H_j | H_{j-1}).$$

Since  $H_j = (H_j^L \text{ or } H_j^R)$  we recognize that the sequence of symbol hits is really a function of the underlying stationary Markov chain  $(H_j^L, H_j^R)$  with four states *viz.*  $(H_j^L, H_j^R) = (0, 0), (0, 1), (1, 0), \text{ or } (1, 1)$ . For the sake of brevity we use  $U_j^{(i)}$  to denote the state of the component channel between the  $i$ -th sender and the corresponding receiver during the  $j$ -th symbol transmission i.e.  $U_j^{(i)} = (H_j^L, H_j^R)$  for the  $i$ -th component channel.

This Markov chain is shown in Fig. 4 and the transition probabilities are calculated in Appendix B. In the  $K$ -user situation each of the  $K$  component channels (between the  $i$ -th sender and the corresponding receiver) is thus a finite state channel. Corresponding to state  $a=(0,0)$  we have a noiseless channel and corresponding to states  $b=(0,1)$ ,  $c=(1,0)$  or  $d=(1,1)$  (i.e. a hit on the current symbol) we have a channel which puts out an erasure symbol.

#### **Case D: Asynchronous Hopping - No Side Information Available**

Finally we address the case where the hopping is asynchronous and the decoders receive no information about whether a symbol is hit or not. Hits are

thus undetected and cause errors with probability  $e_0$ . The component channel in this case may be modelled as an  $M$ -ary input  $M$ -ary output channel where the output letters  $Y_j$  associated with the input  $X_j$  are given by  $Y_j = X_j \oplus H_j R_j$  where  $H_j$  is the random variable (defined earlier) which is 1 if the  $j$ -th symbol was hit and is 0 otherwise and  $R_j$  is a random variable independent of the input and independent of  $H_j$  with a distribution on  $\{0, 1, \dots, M - 1\}$  such that  $P(R_j = 0) = 1 - e_0$  and  $P(R_j = l) = e_0 / (M - 1)$  for  $l = 1, 2, \dots, M - 1$ . Finally the addition  $\oplus$  is modulo  $M$  addition.

### III. Capacity Regions

We observe that in both the synchronous cases our models are a simple case of a  $K$ -user discrete memoryless interference channel, i.e. a channel characterized by a probability density  $p(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(K)} | \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(K)})$  with the  $i$ -th sender trying to communicate with the  $i$ -th receiver through independent encoders and decoders. The capacity region for such channel is not known in general but various inner and outer bounds have been developed for it [6]. Our channels fall into a simple class known as separated channels for which the marginal probabilities  $p(\mathbf{y}^{(i)} | \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(K)})$  do not depend on  $\mathbf{x}^{(j)}$   $j \neq i$ , i.e.

$$p(\mathbf{y}^{(i)} | \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(K)}) = p(\mathbf{y}^{(i)} | \mathbf{x}^{(i)}).$$

Since the capacity region depends only upon the marginal probabilities  $p(\mathbf{y}^{(i)} | \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(K)})$  we see from the converse to the coding theorem for the two user channel that the maximum rate of reliable transmission for the  $i$ th sender-receiver pair cannot be



more that  $\max_{Q_i} I(X_i; Y_i)$  where  $X_i$  and  $Y_i$  are related by the conditional probability distribution  $p(y^{(i)}|x^{(i)})$ . This rate can be actually achieved by maximizing each  $Q_i$  individually and hence we see that the capacity region is

$$0 \leq R_i \leq C_i \quad (3)$$

where  $C_i = \max_{Q_i} I(X_i; Y_i)$ .

In the asynchronous case we similarly see that

$$p(y^{(i)}|x^{(1)}, \dots, x^{(i)}, \dots, x^{(K)}, u^{(1)}, u^{(2)}, \dots, u^{(K)}) = p(y^{(i)}|x^{(i)}, u^{(i)})$$

where the  $u^{(i)}$ 's are the states of the underlying Markov chain corresponding to the  $i$ -th sender-receiver pair. Thus again from the converse to the coding theorem for the single user channel [9] the maximum rate of reliable information transmission cannot be more than the capacity of this finite state two user channel. Since we can actually achieve this rate by suitable choice of input probabilities (in fact, by i.i.d. inputs) we see that the capacity region of this communication system is

$$0 \leq R_i \leq C_i, \quad 1 \leq i \leq K \quad (4)$$

where  $C_i$  is the capacity of the finite state channel corresponding to the  $i$ th sender-receiver pair.

We now calculate the interference capacities for the models described above.

**Case A: (Synchronous Hopping, Side Information Available)**

$$R_i \leq C_i = (1 - \epsilon_A) \log_2 M, \quad i = 1, \dots, K \quad (5)$$

where  $q$  is fixed. The sum of the rates of the individual users,  $R_{sum} = \sum_{i=1}^K R_i$ , is maximized by

$$K^* = \frac{-1}{\ln(1-p)}$$

for which the sum rate is

$$R_{sum} \leq C_{sum} \triangleq \sum_{i=1}^K C_i = \frac{-1}{\ln(1-p)} \frac{e^{-1}}{1-p} \log_2 M.$$

For large  $q$  the optimum number of users approaches  $q$ , i.e.

$$\lim_{q \rightarrow \infty} \frac{T^*}{q} = 1 \quad (6)$$

and

$$\lim_{q \rightarrow \infty} \frac{T^* C_i}{q} = e^{-1} \log_2 M. \quad (7)$$

**Case B: (Synchronous Hopping, No Side Information Available)**

$$R_i \leq C_i = \log_2 M - h_M(\epsilon_B), \quad i = 1, \dots, K \quad (8)$$

and

$$R_{sum} \leq C_{sum} = K(\log_2 M - h_M(\epsilon_B)) \quad (9)$$

where

$$h_M(x) \triangleq -x \log_M(x/(M-1)) - (1-x) \log_M(1-x). \quad (10)$$

The limiting sum capacity for  $M = 2$  and  $K = \lambda q$  with  $\lambda$  fixed is

$$\lim_{q \rightarrow \infty} \frac{C_{sum}}{q} = \lambda(1 - h_2(1 - e^{-\lambda})). \quad (11)$$

Optimizing the above over  $\lambda$  gives  $\lambda^* = 0.46$  and  $\lim_{q \rightarrow \infty} q^{-1}C_{sum} = 0.143$ .

**Case C: (Asynchronous Hopping, Side Information Available)**

The channel model for the  $i$ -th user is a finite state channel with four states  $a, b, c$  and  $d$ . Since the Markov chain  $U_j = (H_j^L, H_j^R)$  is ergodic and its state sequence is independent of the input we see that our channel model belongs to the class of indecomposable finite-state channels with no intersymbol interference memory. We calculate the capacity  $C_i$  of the  $i^{\text{th}}$  component channel as follows (see [9]).

$$C_i = \lim_{n \rightarrow \infty} \max_{Q_n(x^n)} \min_{u_0} \frac{I_Q(X^n; Y^n | u_0)}{n}$$

where  $Q_n(\cdot)$  is some  $n$ -dimensional joint density on the input and  $u_0$  is some initial state of the Markov chain  $(H_j^L, H_j^R)$ . Since the channel is indecomposable and therefore  $C_i$  is independent of the initial state of the Markov chain and may be written as

$$C_i = \lim_{n \rightarrow \infty} \max_{Q_n(x^n)} \frac{I_Q(X^n; Y^n | u_0)}{n}$$

for any initial state  $u_0$ . Now

$$I_Q(X^n; Y^n | u_0) = H(Y^n | u_0) - H(Y^n | X^n, u_0)$$

and

$$H(Y^n|u_0) \leq n(h_M(1 - p(y_j^{(i)} = ?|x_j^{(i)} = k)))$$

with equality achieved by equally inputs where  $h_M(x)$  is given in (10) and  $p(y_j^{(i)} = ?|x_j^{(i)} = k) = 1 - (1 - \frac{1}{q})^{2(K-1)}$  and  $k$  and  $i$  are arbitrary. Denoting  $p(y_j^{(i)} = ?|x_j^{(i)} = k)$  as  $\epsilon_C$ , we have

$$H(Y^n|X^n, u_0) = n[-\epsilon_C \log_M \epsilon_C - (1 - \epsilon_C) \log_M (1 - \epsilon_C)].$$

Now it is easy to verify that

$$C_i = (1 - \epsilon_C) \log_M (M - 1).$$

Hence

$$R_i \leq C_i = \{1 - [1 - (1 - \frac{1}{q})^{2(K-1)}]\} \log_2 M, \quad i = 1, \dots, K \quad (12)$$

is the capacity region of the  $K$ -user channel in this case. Asymptotically as the number of frequency slots,  $q$ , approaches infinity with  $K = \lambda q$ , ( $\lambda$  some constant) we get

$$(1 - (1 - (1 - \frac{1}{q})^{2(K-1)})) \rightarrow 1 - (1 - e^{-2\lambda}) = e^{-2\lambda}. \quad (13)$$

Hence, asymptotically  $C_i = e^{-2\lambda} \log_2 M$ . Optimizing

$$\lim_{q \rightarrow \infty} C_{sum}/q = \lim_{q \rightarrow \infty} \sum_i C_i/q = \lambda e^{-2\lambda} \log_2 M$$

over  $\lambda$  gives  $\lambda^* = 0.5$  and  $\lim_{q \rightarrow \infty} \sum_i C_i/q = 0.5e^{-1} \log_2 M$ .

**Case D: (Asynchronous Hopping, No Side Information Available)**

In this case with asynchronous hopping and no side information available we calculate the capacity  $C_i$  of the  $i^{\text{th}}$  component channel using, as in Case C,

$$C_i = \lim_{n \rightarrow \infty} \max_{Q_n(x^n)} \min_{u_0} \frac{I_Q(X^n; Y^n | u_0)}{n}.$$

Again since the channel is indecomposable  $C_i$  is independent of the initial state of the Markov chain and may be written as

$$C_i = \lim_{n \rightarrow \infty} \max_{Q_n(x^n)} \frac{I_Q(X^n; Y^n | u_0)}{n}$$

for any initial state  $u_0$ . Now

$$\begin{aligned} I_Q(X^n; Y^n | u_0) &= H(Y^n | u_0) - H(Y^n | X^n, u_0) \\ &= H(Y^n | u_0) - H(V^n | u_0) \end{aligned}$$

where  $V_n$  is a 1 if an error occurred on the  $n$ -th symbol and is 0 otherwise. Hence

$$C_i = \lim_{n \rightarrow \infty} \max_{Q_n(x^n)} \frac{I_Q(X^n; Y^n | u_0)}{n} = \log M - H_\infty(V) \quad (13)$$

since  $H(Y^n | u_0) \leq n \log M$  with equality for equally likely outputs which are achieved by equally likely inputs and since  $\lim_{n \rightarrow \infty} \frac{1}{n} H(V^n | u_0)$  is independent of  $u_0$  and is equal to  $H_\infty(V)$ , i. e. the entropy of the stationary random (non-Markovian) process  $\{V_j\}$  which is a function of the Markov chain,  $\{(H_j^L, H_j^R, R_j)\}$ .

Computing the entropy of a function of a Markov process has been considered by Blackwell [11]. While a closed form expression for the capacity is not available in our case, tight upper and lower bounds are available [12]. Let

$W_j = (H_j^L, H_j^R, R_j)$  be the state of a Markov chain and  $V_j$  the function of the state given by  $V_j = 1$  if  $H_j^L = H_j^R = 1$  and  $R_j \neq 0$  and  $V_j = 0$  otherwise. Then the entropy  $H_\infty(V)$  of  $V_j$  is bounded as follows.

$$H(V_n|V_{n-1}, V_{n-2}, \dots, V_1, W_0) \leq H_\infty(V) \leq H(V_n|V_{n-1}, V_{n-2}, \dots, V_1).$$

Furthermore these bounds converge exponentially fast in  $n$  to  $H_\infty(V)$ . From [12] it can be seen that the difference between the upper and lower bounds above is less than  $D\rho^{n-1}$  where

$$D = \frac{N_D \log e}{N_1 \min_{i,j} m_{i,j}},$$

$N_1$  and  $N_D$  are the minimum and maximum number of states collapsed respectively by the function of the Markov chain (in our case  $N_1 = M - 1$ ,  $N_D = 3M + 1$ ),  $m_{i,j}$  is the transition probability of the Markov chain  $W$  and

$$0 < \rho = 1 - \min_{i,j,k,m,n} \frac{N_1 m_{i,k} m_{k,n}}{m_{i,j} m_{j,m}} < 1.$$

In our model this convergence is especially rapid since for most parameters of interest the Markovian dependence of  $W_j$  on  $W_{j-1}$  is very "weak", i.e. the transition probabilities are almost independent of the previous state. In fact our numerical results show that the upper and lower bound are essentially identical for  $n = 2$ , even for moderate values of  $q$ . (For  $q = 50$  the upper and lower bounds agree out to more than 8 significant digits). In Appendix C we show a sample calculation of the entropy used in the upper and lower bounds.

If we let  $q \rightarrow \infty$  with  $K = \lambda q$  ( $\lambda$  constant) we see that the stationary distribution of the Markov chain  $\{U_i\}$  tends to the conditional distribution

$P(U_i|U_{i-1})$  with  $P(U_j = 00) = 1 - e^{-2\lambda}$ ,  $P(U_j = 01) = P(U_j = 10) = e^{-\lambda}(1 - e^{-\lambda})$  and  $P(U_j = 11) = (1 - e^{-\lambda})^2$ . Thus for large  $q$  (and  $K = \lambda q$ ) the  $U_i$  become independent. Thus for the case of errors occurring with probability  $(M - 1)/M$  given a hit occurs, asymptotically as  $K$  and  $q$  become large with  $K/q \rightarrow \lambda$

$$\frac{C_{sum}}{q} = \lambda(\log M - h_M(e^{-2\lambda})). \quad (14)$$

Now it is easy to see that the asymptotic normalized capacity of this asynchronous case is exactly half of the synchronous normalized capacity when optimized over  $\lambda$ .

## IV. Numerical Results and Conclusions

In Fig. 5 we show the sum capacity for the case of synchronous hopping with  $q = 50$  and  $M = 2$  while in Fig. 6 we show the sum capacity for the case of asynchronous hopping with  $q = 50$  and  $M = 2$ . When there is no side information the errors are assumed to occur with probability  $1/2$ . A careful examination of the numerical results show that  $q$  need not be very large for the asymptotic results to give a very accurate approximation to the capacity of these channels. Also the asymptotic value for the optimum number of simultaneous users is a good approximation for the actual value for finite  $q$ . For the asynchronous case without side information the upper and lower bounds were virtually identical for the case of  $q = 50$ . The tightness of the bounds is due to the fact that for even reasonable values of  $q$  the sequence of errors in the channel is essentially an i.i.d. process.

In this paper we have determined the multiple-access capability of frequency-hopped spread-spectrum for four different models. The interference is either modeled as causing errors with a given probability when two users hopped to the same frequency at the same time or as causing erasures. The key result was identifying the underlying Markov chain  $(H_j^L, H_j^R)$  or  $(H_j^L, H_j^R, R_j)$ . Using the Markovian properties of the underlying process allows one to recursively compute the error probabilities of block codes for these channels (see [8]) and for channels with combinations of errors and erasures.



## APPENDIX A

Lemma 1:  $(H_j^L, H_j^R)$  is a Markov chain.

Proof:

$$\begin{aligned}
 & P(H_j^L, H_j^R \mid H_{j-1}^L, H_{j-1}^R, H_{j-2}^L, H_{j-2}^R, H_{j-3}^L, H_{j-3}^R, \dots) \\
 &= \frac{P(H_j^L, H_j^R, H_{j-1}^L, H_{j-1}^R, H_{j-2}^L, H_{j-2}^R, H_{j-3}^L, H_{j-3}^R, \dots)}{P(H_{j-1}^L, H_{j-1}^R, H_{j-2}^L, H_{j-2}^R, \dots)} \\
 &= \frac{A}{B} \quad (\text{say}).
 \end{aligned}$$

We write  $A$  as follows.

$$A = P(F_{1,j} \in_j^L \tilde{F}_{j-1}, F_{1,j} \in_j^R \tilde{F}_j, F_{1,j-1} \in_{j-1}^L \tilde{F}_{j-2}, F_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1} \dots)$$

where  $\in_j^L$  is  $\in$  if  $H_j^L = 1$  and  $\notin$  if  $H_j^L = 0$  and  $\in_j^R$  is  $\in$  if  $H_j^R = 1$  and  $\notin$  if  $H_j^R = 0$ .

Now

$$\begin{aligned}
 A &= \sum_{f_{1,j-1}} P(F_{1,j} \in_j^R \tilde{F}_{j-1}, F_{1,j} \in_j^R \tilde{F}_j, F_{1,j-1} \in_j^L \tilde{F}_{j-2}, F_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1}, \dots \mid F_{1,j-1} = f_{1,j-1}) \\
 &\quad \cdot P(F_{1,j-1} = f_{1,j-1}) \\
 &= \sum_{f_{1,j-1}} P(F_{1,j} \in_j^L \tilde{F}_{j-1}, F_{1,j} \in_j^R \tilde{F}_j, f_{1,j-1} \in_j^L \tilde{F}_{j-2}, f_{1,j-1} \in_j^R \tilde{F}_{j-1} \dots) \frac{1}{q} \\
 &= \sum_{f_{1,j-1}} P(F_{1,j} \in_j^L \tilde{F}_{j-1}, F_{1,j} \in_j^R \tilde{F}_j, f_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1}) \\
 &\quad \cdot P(f_{1,j-1} \in_{j-1}^L \tilde{F}_{j-2}, f_{1,j-1} \in_{j-2}^L \tilde{F}_{j-1} \dots) \frac{1}{q}
 \end{aligned}$$

using the independence of  $F_{1,j}, F_{1,j-1}, F_{1,j-2}, \tilde{F}_j, \tilde{F}_{j-1}, \tilde{F}_{j-2}, \dots$  in the last equality.

Now from the fact the  $F_{i,j}$  for each user  $i$  is a random variable uniformly distributed over the set  $\{1, \dots, q\}$  and  $F_{i,j}$  and  $F_{k,j}$  are independent and identically

distributed for  $i \neq k$  we see that  $P(F_{1,j} \in_j^L \tilde{F}_{j-1}, F_{1,j} \in_j^R \tilde{F}_j, f_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1})$  is functionally independent of  $f_{1,j-1}$ . Thus

$$\begin{aligned}
& P(F_{1,j} \in_j^L \tilde{F}_{j-1}, F_{1,j} \in_j^R \tilde{F}_j, f_{1,j-1} \in_{j-1}^L \tilde{F}_{j-1}) \\
& \quad \sum_{f_{1,j-1}} \frac{1}{q} P(f_{1,j-1} \in_{j-1}^L F_{j-2}, F_{1,j-2} \in_{j-2}^L \tilde{F}_{j-3} \dots) \\
& = P(F_{1,j} \in_j^L \tilde{F}_{j-1}, F_{1,j} \in_j^R \tilde{F}_j, f_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1}) \\
& \quad P(F_{1,j-1} \in_{j-1}^L F_{j-2}, F_{1,j-2} \in_{j-2}^L \tilde{F}_{j-3} \dots).
\end{aligned}$$

Similarly it can be seen that

$$B = P(f_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1}) P(F_{1,j-1} \in_{j-1}^L F_{j-2}, F_{1,j-2} \in_{j-2}^L F_{j-3} \dots)$$

and so

$$\begin{aligned}
\frac{A}{B} & = \frac{P(F_{1,j} \in_{j-1}^L \tilde{F}_{j-1}, F_{1,j} \in_j^R F_j, f_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1})}{P(f_{1,j-1} \in_{j-1}^R \tilde{F}_{j-1})} \\
& = P(H_j^L, H_j^R \mid H_{j-1}^R).
\end{aligned}$$

Clearly  $P(H_j^L, H_j^R \mid H_{j-1}^L, H_{j-1}^R)$  will also be equal to  $P(H_j^L, H_j^R \mid H_{j-1}^R)$ . Thus the Lemma follows.

## APPENDIX B

In this appendix we calculate the stationary probability distribution and transition probabilities for the Markov chain shown in Fig. 5. We will show how to calculate the stationary probability for one particular state. The calculations for the other states are similar and so we just state the result.

$$\begin{aligned}
 P\{H_j^L = 0, H_j^R = 0\} &= P\{F_{1,j} \notin \tilde{F}_{j-1}, F_{1,j} \notin \tilde{F}_j\} \\
 &= \sum_{f_{1,j}} P\{f_{1,j} \notin \tilde{F}_{j-1}, f_{1,j} \notin \tilde{F}_j | F_{1,j} = f_{1,j}\} P\{F_{1,j} = f_{1,j}\} \\
 &= \frac{1}{q} \sum_{f_{1,j}} P\{f_{1,j} \notin \tilde{F}_{j-1} | F_{1,j} = f_{1,j}\} P\{f_{1,j} \notin \tilde{F}_j | F_{1,j} = f_{1,j}\} \\
 &= \frac{1}{q} \sum_{f_{1,j}} \left(1 - \frac{1}{q}\right)^{K-1} \left(1 - \frac{1}{q}\right)^{K-1} \\
 &= \left(1 - \frac{1}{q}\right)^{2(K-1)}.
 \end{aligned}$$

Similarly

$$P\{H_j^L = 0, H_j^R = 1\} = \left(1 - \frac{1}{q}\right)^{K-1} \left[1 - \left(1 - \frac{1}{q}\right)^{K-1}\right],$$

$$P\{H_j^L = 1, H_j^R = 0\} = \left(1 - \frac{1}{q}\right)^{K-1} \left[1 - \left(1 - \frac{1}{q}\right)^{K-1}\right],$$

$$P\{H_j^L = 1, H_j^R = 1\} = \left[1 - \left(1 - \frac{1}{q}\right)^{K-1}\right]^2.$$

The transition probabilities of the Markov chain are easily determined once  $q(i, k) = P\{H_j^L = i, H_{j-1}^R = k\}$  is determined. Let  $\alpha = \left(1 - \frac{1}{q}\right)^{K-1}$  and

$\beta = (1 - \frac{2}{q})^{K-1}$ . Then

$$\begin{aligned}
q(0,0) &= P\{H_j^L = 0, H_{j-1}^R = 0\} \\
&= \frac{1}{q^2} \sum_{f_{1,j}, f_{1,j-1}} P\{f_{1,j} \notin \tilde{F}_{j-1}, f_{1,j-1} \notin \tilde{F}_{j-1}\} \\
&= \frac{1}{q^2} [q\alpha + q(q-1)\beta] \\
&= \frac{1}{q}\alpha + (1 - \frac{1}{q})\beta,
\end{aligned}$$

$$\begin{aligned}
q(1,1) = P\{H_j^L = 1, H_{j-1}^R = 1\} &= \frac{1}{q^2} \sum_{f_{1,j}, f_{1,j-1}} P\{f_{1,j} \in \tilde{F}_{j-1}, f_{1,j-1} \in \tilde{F}_{j-1}\} \\
&= \frac{1}{q^2} [q(1-\alpha) + q(q-1)\{2(1-\alpha) - (1-\beta)\}] \\
&= \frac{1}{q}(1-\alpha) + (1 - \frac{1}{q})[2(1-\alpha) - (1-\beta)].
\end{aligned}$$

Due to symmetry it is easy to see that  $q(0,1) = q(1,0)$  and thus

$$\begin{aligned}
q(0,1) = q(1,0) &= (1 - q(0,0) - q(1,1))/2 \\
&= (1 - \frac{1}{q})(\alpha - \beta).
\end{aligned}$$

We now turn to the calculation of the transition probabilities for the Markov chain.

$$P\{H_j^L = l, H_j^R = m | H_{j-1}^R = n\} = \frac{P\{H_j^L = l, H_j^R = m, H_{j-1}^R = n\}}{P\{H_{j-1}^R = n\}}.$$

For the case  $l = 0, m = 1, n = 0$  the numerator in the above expression can be written as

$$P\{H_j^L = l, H_j^R = m, H_{j-1}^R = n\} = \frac{1}{q^2} \sum_{f_{1,j}, f_{1,j-1}} P\{f_{1,j} \notin \tilde{F}_{j-1}, f_{1,j-1} \notin \tilde{F}_{j-1}\} P\{f_{1,j} \in \tilde{F}_j\}.$$

Since the last term does not depend on  $f_{1,j}$  this term can be taken out of the summation we have

$$P\{H_j^L = 0, H_j^R = 1, H_{j-1}^R = 0\} = P\{H_j^L = 0, H_{j-1}^R = 1\}P\{H_j^R = 0\}.$$

It is easy to see that these operations also hold for any value  $l, m, n$  so we have in general that

$$P\{H_j^L = l, H_j^R = m, H_{j-1}^R = n\} = P\{H_j^L = l, H_{j-1}^R = m\}P\{H_j^R = n\}.$$

Thus

$$\begin{aligned} p(l, m|n) &\triangleq P\{H_j^L = l, H_j^R = m | H_{j-1}^R = n\} \\ &= \frac{P\{H_j^L = l, H_{j-1}^R = n\}P\{H_j^R = m\}}{P\{H_{j-1}^R = n\}} \\ &= \frac{q(l, n)p(m)}{p(n)} \end{aligned}$$

where  $p(k) = P\{H_j^R = k\}$ . Letting  $a = (0, 0)$ ,  $b = (0, 1)$ ,  $c = (1, 0)$  and  $d = (1, 1)$  and  $p_{i,l} = P(H_j^L, H_j^R = i | (H_{j-1}^L, H_{j-1}^R) = l)$ , the transition probabilities, are given as

$$\begin{aligned} p_{a,a} = p_{a,c} = p(0, 0|0) &= \frac{1}{q}\alpha + (1 - \frac{1}{q})\beta, \\ p_{b,a} = p_{b,c} = p(0, 1|0) &= [\frac{1}{q}\alpha + (1 - \frac{1}{q})\beta](\frac{1 - \alpha}{\alpha}), \\ p_{c,a} = p_{c,c} = p(1, 0|0) &= (1 - \frac{1}{q})(\alpha - \beta), \\ p_{d,a} = p_{d,c} = p(1, 1|0) &= (1 - \frac{1}{q})(\alpha - \beta)(\frac{1 - \alpha}{\alpha}), \\ p_{a,b} = p_{a,d} = p(0, 0|1) &= (1 - \frac{1}{q})(\alpha - \beta)(\frac{\alpha}{1 - \alpha}), \end{aligned}$$

$$\begin{aligned}
p_{b,b} = p_{b,d} = p(0,1|1) &= \left(1 - \frac{1}{q}\right)(\alpha - \beta), \\
p_{c,b} = p_{c,d} = p(1,0|1) &= \left[\frac{1}{q}(1 - \alpha) + \left(1 - \frac{1}{q}\right)(2(1 - \alpha) - (1 - \beta))\right]\left(\frac{\alpha}{1 - \alpha}\right), \\
p_{d,b} = p_{d,d} = p(1,1|1) &= \frac{1}{q}(1 - \alpha) + \left(1 - \frac{1}{q}\right)[2(1 - \alpha) - (1 - \beta)].
\end{aligned}$$

## APPENDIX C

In this Appendix we give a few sample calculations for the joint distributions of the random variable  $V_j$ , a function of the Markov chain  $W_j = (H_j^L, H_j^R, R_j)$ . That this is a Markov chain is an easy consequence of the fact that  $(H_j^L, H_j^R)$  is a Markov chain and is independent of  $\{R_j\}$  which is an i.i.d. process.

First let  $U_j = (H_j^L, H_j^R)$  with  $a = (0, 0)$ ,  $b = (0, 1)$ ,  $c = (1, 0)$ , and  $d = (1, 1)$ . The upper bound to the entropy of  $V_j$  is determined from the joint distribution of  $V_j$ . We will do a sample calculation of the joint distribution for one particular argument for  $n = 2$  and  $n = 3$  and list the results for other arguments. The distributions of  $V_j$  will be calculated in terms of the distributions of the Markov chain  $U_j$  and the joint distribution of the process  $\{R_j\}$ . Since  $R_j$  is an i.i.d. process the joint distribution is easy to calculate. The joint distribution of  $U_j$  is easily calculated since it is a Markov chain.

The first order distribution of  $V_j$  is calculated as follows.

$$\begin{aligned} P(V_0 = 0) &= P(H_0 = 0 \text{ or } R_0 = 0) = P(H_0 = 0) + P(H_0 = 1)P(R_0 = 0) \\ &= P(U_0 = a) + P(U_0 \in B)P(R_0 = 0), \end{aligned}$$

$$\begin{aligned} P(V_0 = 1) &= P(H_0 = 1, R_0 = 1) = P(H_0 = 1)P(R_0 = 1) \\ &= P(U_0 \in B)P(R_0 = 1). \end{aligned}$$

The second order distribution is calculated as follows

$$\begin{aligned}
P(V_1 = 0, V_0 = 0) &= P(H_1 = 0 \text{ or } R_1 = 0, H_0 = 0 \text{ or } R_0 = 0) \\
&= P(H_1 = 0, H_0 = 0 \text{ or } R_0 = 0) \\
&\quad + P(H_1 = 1, R_1 = 0, H_0 = 0 \text{ or } R_0 = 0) \\
&= P(H_1 = 0, H_0 = 0) + P(H_1 = 0, H_0 = 1, R_0 = 0) \\
&\quad + P(H_1 = 1, R_1 = 0, H_0 = 0) + P(H_1 = 1, R_1 = 0, H_0 = 1, R_0 = 0) \\
&= P(H_1 = 0, H_0 = 0) + P(H_1 = 0, H_0 = 1)P(R_0 = 0) \\
&\quad + P(H_1 = 1, H_0 = 0)P(R_1 = 0) + P(H_1 = 1, H_0 = 1)P(R_1 = 0, R_0 = 0) \\
&= P(U_1 = a, U_0 = a) + P(U_1 = a, U_0 \in B)P(R_0 = 0) \\
&\quad + P(U_1 \in B, U_0 = a)P(R_1 = 0) + P(U_1 \in B, U_0 \in B)P(R_1 = 0, R_0 = 0).
\end{aligned}$$

Later we determine the joint distribution of  $(U_1, U_0)$  that will allow us to complete the calculation. The remaining components of the distribution are calculated in a similar fashion and so we just state the results.

$$\begin{aligned}
P(V_1 = 0, V_0 = 1) &= P(H_1 = 0 \text{ or } R_1 = 0, H_0 = 1, R_0 = 1) \\
&= P(U_1 = a, U_0 \in B)P(R_0 = 1) + P(U_1 \in B, U_0 \in B)P(R_1 = 0)P(R_0 = 1) \\
P(V_1 = 1, V_0 = 0) &= P(H_1 = 1, R_1 = 1, H_0 = 0 \text{ or } R_0 = 0) \\
&= P(U_1 \in B, U_0 = a)P(R_1 = 1) + P(U_1 \in B, U_0 \in B)P(R_1 = 1)P(R_0 = 0)
\end{aligned}$$

$$P(V_1 = 1, V_0 = 1) = 1 - P(V_1 = 1, V_0 = 0) - P(V_1 = 0, V_0 = 1) - P(V_1 = 0, V_0 = 0).$$



For  $n = 3$  the joint distribution is calculated using the same method as for  $n = 2$ .

We obtain

$$\begin{aligned}
P(V_2 = 0, V_1 = 0, V_0 = 0) &= P(H_2 = 0 \text{ or } R_2 = 0, H_1 = 0 \text{ or } R_1 = 0, H_0 = 0 \text{ or } R_0 = 0) \\
&= P(U_2 = a, U_1 = a, U_0 = a) \\
&\quad + P(U_2 = a, U_1 \in B, U_0 = a)P(R_1 = 0) \\
&\quad + P(U_2 = a, U_1 = a, U_0 \in B)P(R_0 = 0) \\
&\quad + P(U_2 = a, U_1 \in B, U_0 \in B)P(R_1 = 0, R_0 = 0) \\
&\quad + P(U_2 \in B, U_1 = a, U_0 = a)P(R_2 = 0) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 = a)P(R_2 = 0, R_1 = 0) \\
&\quad + P(U_2 \in B, U_1 = a, U_0 \in B)P(R_2 = 0, R_0 = 0) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 0, R_1 = 0, R_0 = 0),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 1, V_1 = 0, V_0 = 0) &= P(H_2 = 1, R_2 = 1, H_1 = 0 \text{ or } R_1 = 0, H_0 = 0 \text{ or } R_0 = 0) \\
&= P(U_2 \in B, U_1 = a, U_0 = a)P(R_2 = 1) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 = a)P(R_2 = 1, R_1 = 0) \\
&\quad + P(U_2 \in B, U_1 = a, U_0 \in B)P(R_2 = 1, R_0 = 0) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 1, R_1 = 0, R_0 = 0),
\end{aligned}$$

$$P(V_2 = 0, V_1 = 1, V_0 = 0) = P(H_2 = 0 \text{ or } R_2 = 0, H_1 = 1, R_1 = 1, H_0 = 0 \text{ or } R_0 = 0)$$

$$\begin{aligned}
&= P(U_2 = a, U_1 \in B, U_0 = a)P(R_1 = 1) \\
&\quad + P(U_2 = a, U_1 \in B, U_0 \in B)P(R_1 = 1, R_0 = 0) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 = a)P(R_2 = 0, R_1 = 1) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 0, R_1 = 1, R_0 = 0),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 1, V_1 = 1, V_0 = 0) &= P(H_2 = 1, R_2 = 1, H_1 = 1, R_1 = 1, H_0 = 0 \text{ or } R_0 = 0) \\
&= P(U_2 \in B, U_1 \in B, U_0 = a)P(R_2 = 1, R_1 = 1) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 1, R_1 = 1, R_0 = 0),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 0, V_1 = 0, V_0 = 1) &= P(H_2 = 0 \text{ or } R_2 = 0, H_1 = 0 \text{ or } R_1 = 0, H_0 = 1, R_0 = 1) \\
&= P(U_2 = a, U_1 = a, U_0 \in B)P(R_0 = 1) \\
&\quad + P(U_2 = a, U_1 \in B, U_0 \in B)P(R_1 = 0, R_0 = 1) \\
&\quad + P(U_2 \in B, U_1 = a, U_0 \in B)P(R_2 = 0, R_0 = 1) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 0, R_1 = 0, R_0 = 1),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 1, V_1 = 0, V_0 = 1) &= P(H_2 = 1, R_2 = 1, H_1 = 0 \text{ or } R_1 = 0, H_0 = 1, R_0 = 1) \\
&= P(U_2 \in B, U_1 = a, U_0 \in B)P(R_2 = 1, R_0 = 1) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 1, R_1 = 0, R_0 = 1),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 0, V_1 = 1, V_0 = 1) &= P(H_2 = 0 \text{ or } R_2 = 0, H_1 = 1, R_1 = 1, H_0 = 1, R_0 = 1) \\
&= P(U_2 = a, U_1 \in B, U_0 \in B)P(R_1 = 1, R_0 = 0) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 0, R_1 = 1, R_0 = 1),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 1, V_1 = 1, V_0 = 1) &= P(H_2 = 1, R_2 = 1, H_1 = 1, R_1 = 1, H_0 = 1, R_0 = 1) \\
&= P(U_2 \in B, U_1 \in B, U_0 \in B)P(R_2 = 1, R_1 = 1, R_0 = 0).
\end{aligned}$$

Now since  $U_j$  is a Markov chain [10]

$$\begin{aligned}
P(U_2 = a, U_1 = a, U_0 = a) &= P(U_2 = a|U_1 = a)P(U_1 = a|U_0 = a)P(U_0 = a) \\
P(U_2 = a, U_1 = a, U_0 \in B) &= P(U_2 = a|U_1 = a)P(U_1 = a|U_0 \in B)P(U_0 \in B) \\
P(U_2 = a, U_1 \in B, U_0 = a) &= \sum_{\gamma \in B} P(U_2 = a|U_1 = \gamma)P(U_1 = \gamma|U_0 = a)P(U_0 = a) \\
P(U_2 = a, U_1 \in B, U_0 \in B) &= \sum_{\gamma \in B} P(U_2 = a|U_1 = \gamma)P(U_1 = \gamma|U_0 \in B)P(U_0 \in B) \\
P(U_2 \in B, U_1 = a, U_0 = a) &= P(U_2 \in B|U_1 = a)P(U_1 = a|U_0 = a)P(U_0 = a) \\
P(U_2 \in B, U_1 = a, U_0 \in B) &= P(U_2 \in B|U_1 = a)P(U_1 = a|U_0 \in B)P(U_0 \in B) \\
P(U_2 \in B, U_1 \in B, U_0 = a) &= \sum_{\gamma \in B} P(U_2 \in B|U_1 = \gamma)P(U_1 = \gamma|U_0 = a)P(U_0 = a) \\
P(U_2 \in B, U_1 \in B, U_0 \in B) &= \sum_{\gamma \in B} P(U_2 \in B|U_1 = \gamma)P(U_1 = \gamma|U_0 \in B)P(U_0 \in B).
\end{aligned}$$

The transition probabilities above are calculated in terms of the transition probabilities of the Markov chain  $U_j$  which are determined in Appendix B.

$$P(U_j = a|U_{j-1} = a) = p_{a,a}$$

$$\begin{aligned}
P(U_j = a|U_{j-1} \in B) &= P(U_j = a, U_{j-1} \in B) / P(U_j \in B) \\
&= \frac{[P(U_j = a, U_{j-1} = b) + P(U_j = a, U_{j-1} = c) + P(U_j = a, U_{j-1} = d)]}{P(U_{j-1} = b) + P(U_{j-1} = c) + P(U_{j-1} = d)} \\
&= \frac{p_{a,b}p_b + p_{a,c}p_c + p_{a,d}p_d}{p_b + p_c + p_d} \\
P(U_j \in B|U_{j-1} = a) &= P(U_j = b|U_{j-1} = a) + P(U_j = c|U_{j-1} = a) + P(U_j = d|U_{j-1} = a) \\
&= p_{b,a} + p_{c,a} + p_{d,a}
\end{aligned}$$

for any integer  $j$ .

Now consider the lower bound to the entropy, namely,

$$H(V_n|V_{n-1}, \dots, V_1, W_0).$$

Since  $R_0$  is independent of  $V_j$  for all  $j$  it is easy to see that this entropy is the same as

$$H(V_n|V_{n-1}, \dots, V_1, U_0).$$

To calculate this entropy we need to know the joint distribution of  $(V_n, \dots, V_1, U_0)$ . The bound we use will be with  $n = 2$ . The second order distribution is given as follows.

$$\begin{aligned}
P(V_2 = 0, V_1 = 0, U_0 = i) &= P(H_2 = 0 \text{ or } R_2 = 0, H_1 = 0 \text{ or } R_1 = 0, U_0 = i) \\
&= P(H_2 = 0, H_1 = 0, U_0 = i) \\
&\quad + P(H_2 = 0, H_1 = 1, R_1 = 0, U_0 = i) \\
&\quad + P(H_2 = 1, R_2 = 0, H_1 = 0, U_0 = i) \\
&\quad + P(H_2 = 1, R_2 = 0, H_1 = 1, R_1 = 0, U_0 = i)
\end{aligned}$$

$$\begin{aligned}
&= P(H_2 = 0, H_1 = 0, U_0 = i) \\
&\quad + P(H_2 = 0, H_1 = 1, U_0 = i)P(R_1 = 0) \\
&\quad + P(H_2 = 1, H_1 = 0, U_0 = i)P(R_2 = 0) \\
&\quad + P(H_2 = 1, H_1 = 1, U_0 = i)P(R_2 = 0, R_1 = 0) \\
&= P(U_2 = a, U_1 = a, U_0 = i) \\
&\quad + P(U_2 = a, U_1 \in B, U_0 = i)P(R_1 = 0) \\
&\quad + P(U_2 \in B, U_1 = a, U_0 = i)P(R_2 = 0) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 = i)P(R_2 = 0, R_1 = 0),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 1, V_1 = 0, U_0 = i) &= P(H_2 = 1, R_2 = 1, H_1 = 0 \text{ or } R_1 = 0, U_0 = i) \\
&= P(U_2 \in B, U_1 = a, U_0 = i)P(R_2 = 1) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 = i)P(R_2 = 1, R_1 = 0),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 0, V_1 = 1, U_0 = i) &= P(H_2 = 0 \text{ or } R_2 = 0, H_1 = 1, R_1 = 1, U_0 = i) \\
&= P(U_2 = a, U_1 \in B, U_0 = i)P(R_1 = 1) \\
&\quad + P(U_2 \in B, U_1 \in B, U_0 = i)P(R_2 = 0, R_1 = 1),
\end{aligned}$$

$$\begin{aligned}
P(V_2 = 1, V_1 = 1, U_0 = i) &= P(H_2 = 1, R_2 = 1, H_1 = 1, R_1 = 1, U_0 = i) \\
&= P(U_2 \in B, U_1 \in B, U_0 = i)P(R_2 = 1, R_1 = 1).
\end{aligned}$$

## References

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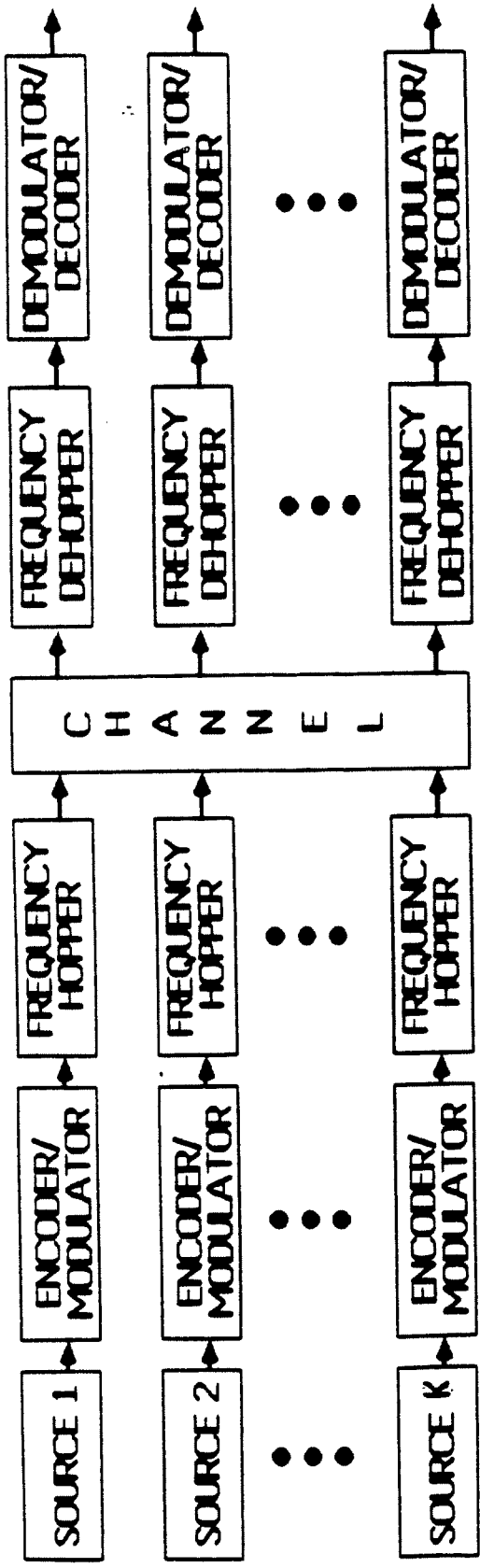


Figure 1. Multi-user frequency-hop spread-spectrum communication system.





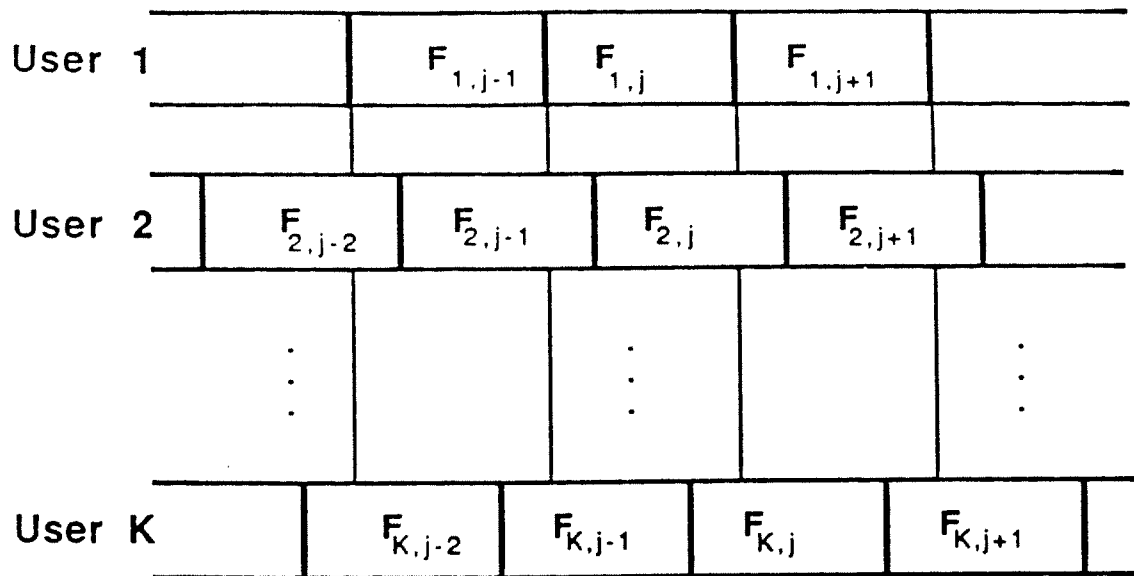


Fig. 3. Timing diagram for frequency-hopped multiple-access

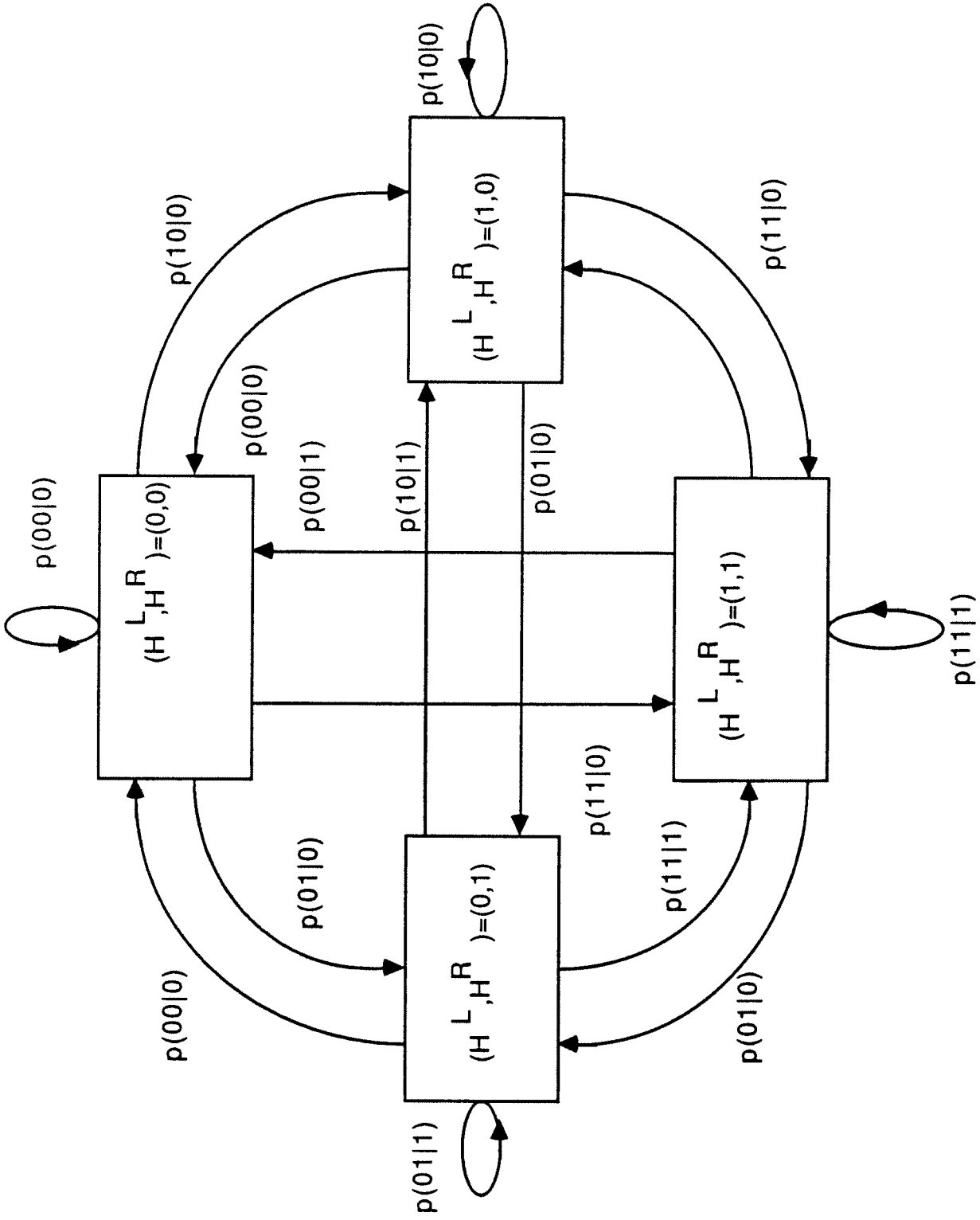


Fig. 4. Markov chain for frequency-hopped spread-spectrum multiple-access communications.

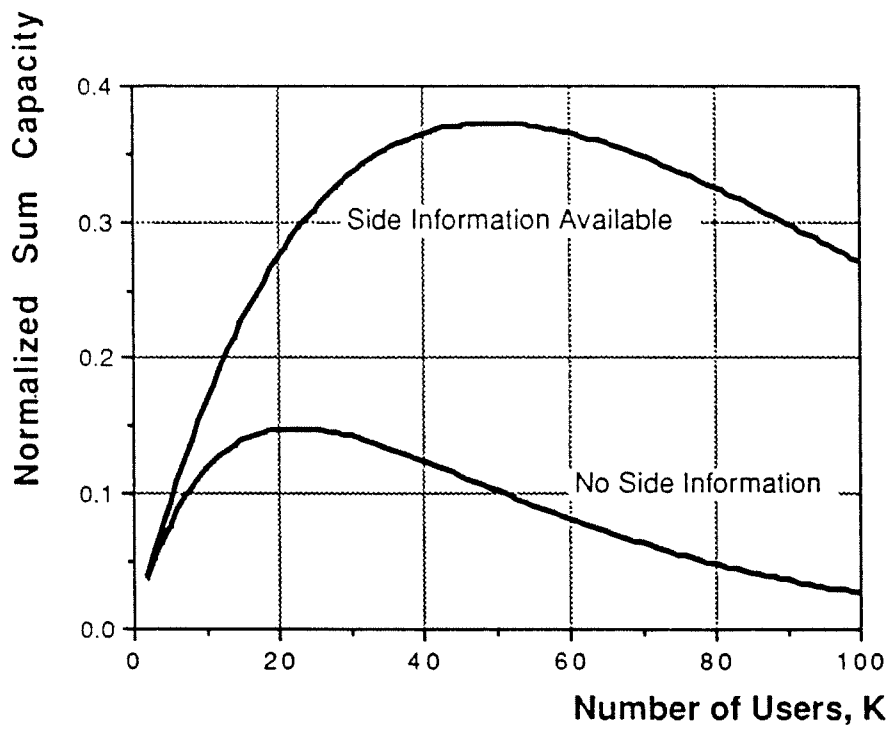


Fig. 5. Normalized sum capacity for synchronous hopping with  $q = 50$ .

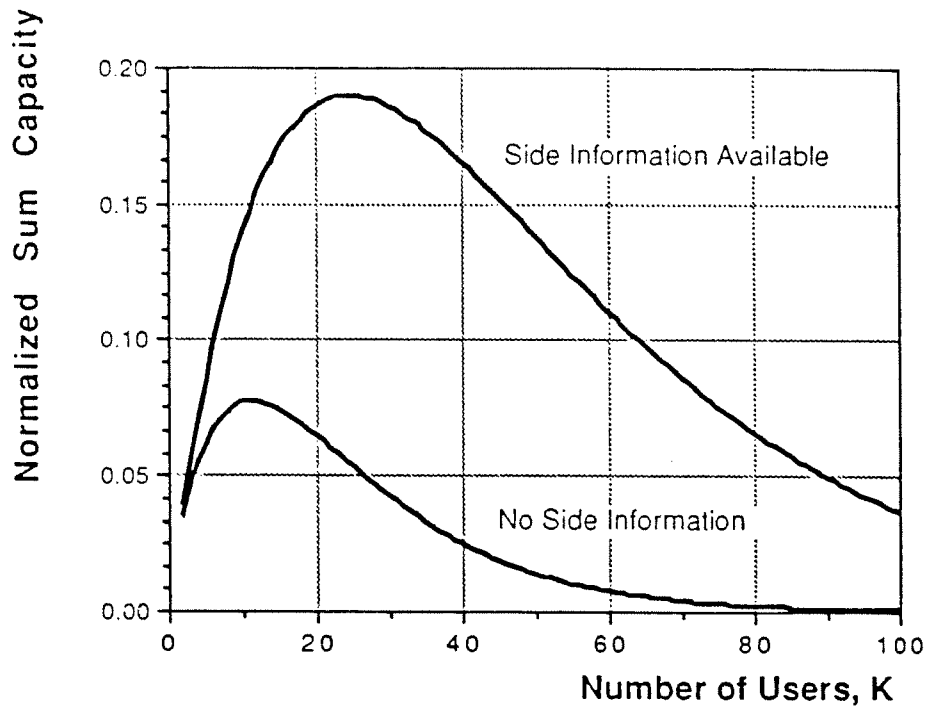


Fig. 6. Normalized sum capacity for asynchronous hopping with  $q = 50$ .