On Transformations of Random Vectors

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Abstract

This technical report treats some technical considerations related to the probability density function of a function of a random vector.

I. Introduction

Let $X \in R^n$ be a continuous random vector with known pdf $f_X(x)$. In many problems it is necessary to find the pdf $f_Y(y)$ of a new random vector Y defined as a function $Y = g(X)$ of X, where $g : R^n \to R^n$.

Many textbooks on probability and random variables state the following equality:

$$f_Y(y) = f_X(g^{-1}(y)) \, |J(y)|, \tag{1}$$

where J is the Jacobian of $g^{-1}(y)$.

There is considerable variation in how precisely the textbook authors state the conditions for the above equality. Most books do state the condition that g be one-to-one (and hence invertible). However, the stated conditions on differentiability vary widely.

Many engineering books make *no mention* of the need for g^{-1} to be differentiable, *e.g.* [1–7]. Many books assume that g^{-1} is *globally* differentiable *e.g.* [8–13], but this condition is too restrictive in some applications. Some books [14–16] assume that $g : \mathcal{S} \to g(\mathcal{S})$ is one-to-one and differentiable on some open set $\mathcal{S} \subseteq R^n$, and that the pdf of X vanishes (is zero) outside of $\mathcal{S}$. This is reasonably general, but still inapplicable to problems where, for example, X has a Gaussian pdf and $\mathcal{S}$ is a proper subset of R^n.

A more general requirement is to assume that $P[X \in \mathcal{S}] = 1$, for which the condition that $f_X(x)$ vanishes outside $\mathcal{S}$ is a special case. Hoel, Port, and Stone [17] provide such a theorem without proof. Bickel and Doksum [18] provide a proof of the transformation formula under the condition $P[X \in \mathcal{S}] = 1$, but the proof is not entirely rigorous since the integrals given can cover points outside $\mathcal{S}$ where the Jacobian need not exist. This technical report provides a rigorous proof of (1), properly handling the technical details of the set $\mathcal{S}$.

This work was motivated by [19], in which a transformation function arises that is differentiable except on a set of hyperplanes of Lebesgue measure zero.

II. Theory

The following is simply Theorem 17.2 of [14], included for convenience.

Theorem 1: Let $h : \mathcal{V} \to h(\mathcal{V})$ be a one-to-one mapping of an open set $\mathcal{V}$ onto an open set $h(\mathcal{V})$. Suppose that (on $\mathcal{V}$) h is continuous and that h has continuous partial derivatives h_{ij} with Jacobian $J(y) \triangleq \det[h_{ij}(y)]$. Then for $\mathcal{A} \subseteq \mathcal{V}$, for any nonnegative function f

$$\int_{\mathcal{A}} f(h(y)) |J(y)| \, dy = \int_{h(\mathcal{A})} f(x) \, dx. \tag{2}$$

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The following Theorem is a generalization of (20.20) in [14]. Standard treatments *e.g.* [13, p. 143] assume that the transformation function is globally differentiable. Our generalization allows for a (measure zero) set where the Jacobian is undefined.

Theorem 2: Let $g : R^n \to R^n$ be one-to-one and assume that $h = g^{-1}$ is continuous. Assume that on an open set $\mathcal{V} \subseteq R^n$ h is continuously differentiable with Jacobian $J(y)$. Define $J_0 : R^n \to \mathbb{R}$ by

$$J_0(y) = \begin{cases} J(y), & y \in \mathcal{V} \\ 0, & y \in \mathcal{V}^c, \end{cases} \tag{3}$$

where $\mathcal{V}^c$ is the set complement (in R^n) of $\mathcal{V}$.

Define $\mathcal{U} = h(\mathcal{V})$. Suppose random vector X has pdf $f_X(x)$ (with respect to Lebesgue measure) with nonzero mass in $\mathcal{U}^c$, *i.e.* $P[X \in \mathcal{U}^c] = \int_{\mathcal{U}^c} f_X(x)\,dx = 0$. Then the pdf of $Y = g(X)$ is given by

$$\boxed{f_Y(y) = f_X(g^{-1}(y))\,|J_0(y)| = \begin{cases} f_X(g^{-1}(y))\,|J(y)|\,, & y \in \mathcal{V} \\ 0, & y \in \mathcal{V}^c. \end{cases}} \tag{4}$$

Proof:
For (measurable) $\mathcal{B} \subseteq R^n$

$$0 \leq P[g(X) \in \mathcal{B} \cap \mathcal{V}^c] \leq P[g(X) \in \mathcal{V}^c] = P[X \in g^{-1}(\mathcal{V}^c)] = P[X \in \mathcal{U}^c] = 0.$$

Thus $P[g(X) \in \mathcal{B} \cap \mathcal{V}^c] = 0$, so

$$\begin{aligned} P[g(X) \in \mathcal{B}] &= P[g(X) \in \mathcal{B} \cap \mathcal{V}] + P[g(X) \in \mathcal{B} \cap \mathcal{V}^c] \\ &= P[X \in h(\mathcal{B} \cap \mathcal{V})] = \int_{h(\mathcal{B} \cap \mathcal{V})} f_X(x)\,dx = \int_{\mathcal{B} \cap \mathcal{V}} f_X(h(y))\,|J(y)|\;dy \end{aligned}$$

by Theorem 1, which applies since $\mathcal{B} \cap \mathcal{V} \subseteq \mathcal{V}$. (The set $\mathcal{U}$ is open since by assumption $\mathcal{V}$ is open and h is continuous.) Thus by (3):

$$P[g(X) \in \mathcal{B}] = \int_{\mathcal{B} \cap \mathcal{V}} f_X(h(y))\,|J_0(y)|\;dy = \int_{\mathcal{B}} f_X(h(y))\,|J_0(y)|\;dy - \int_{\mathcal{B} \cap \mathcal{V}^c} f_X(h(y))\,|J_0(y)|\;dy,$$

since $\mathcal{B}$ is the union of the disjoint sets $\mathcal{B} \cap \mathcal{V}$ and $\mathcal{B} \cap \mathcal{V}^{\,c}$. The second integral above is zero since $|J_0(y)|$ is zero for $y \in \mathcal{V}^c$ by (3). Thus

$$P[g(X) \in \mathcal{B}] = \int_{\mathcal{B}} f_X(h(y))\,|J_0(y)|\;dy,$$

for $\mathcal{B} \subseteq R^n$, proving that (4) is a pdf of $g(X)$. □

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