Asymptotic Connectivity of Low Duty-Cycled Wireless Sensor Networks

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Abstract*— In this report we study the asymptotic connectivity of a low duty-cycled wireless sensor network, where all sensors are randomly duty-cycled such that they are on/active at any time with a fixed probability. A wireless network is often said to be asymptotically connected if there exists a path from every node to every other node in the network with high probability as the network density approaches infinity. Within the context of a low duty-cycled wireless sensor network, the network is said to be asymptotically connected if for all realizations of the random duty-cycling (i.e., the combination of on and off nodes) there exists a path of active nodes from every node to every other node in the network with high probability as the network density approaches infinity. With this definition, we derive conditions under which a low duty-cycled sensor network is asymptotically connected. These conditions essentially specify how the nodes' communication range and the duty-cycling probability should scale as the network grows in order to maintain connectivity.*In this report we study the asymptotic connectivity of a low duty-cycled wireless sensor network, where all sensors are randomly duty-cycled such that they are on/active at any time with a fixed probability. A wireless network is often said to be asymptotically connected if there exists a path from every node to every other node in the network with high probability as the network density approaches infinity. Within the context of a low duty-cycled wireless sensor network, the network is said to be asymptotically connected if for all realizations of the random duty-cycling (i.e., the combination of on and off nodes) there exists a path of active nodes from every node to every other node in the network with high probability as the network density approaches infinity. With this definition, we derive conditions under which a low duty-cycled sensor network is asymptotically connected. These conditions essentially specify how the nodes' communication range and the duty-cycling probability should scale as the network grows in order to maintain connectivity.**

I. Introduction

The Army's Future Combat Systems potentially rely heavily on the efficient use of unattended sensors to detect, identify and track targets in order to enhance situation awareness, agility and survivability. Among different types of sensors, the unattended ground sensors (UGS) are typically deployed and left to self-organize and carry out various sensing, monitoring, surveillance and communication tasks. These sensors are operated on battery power, and energy is not always renewable due to cost, environmental and form-size concerns. This imposes a stringent energy constraint on the design of the communication architecture, communication protocols, and the deployment and operation of these sensors. It is thus critical to operate these sensors in a highly energy efficient manner.

It has been observed that low power sensors consume significant amount of energy while idling in addition to that consumed during transmission and reception. Consequently, it has been widely considered a key method of energy conservation to turn off sensors that are not actively involved in sensing or communication. By functioning at a *low duty cycle*, i.e., by reducing the fraction of time that a sensor is active/on, sensors are able to conserve energy, which consequently leads to prolonged lifetime. This is particularly applicable in scenarios where sensors are naturally idle for most of the time (e.g., detection of infrequent events such as fire, fault, etc., and transmission of very short messages). However, as sensors alternate between sleep and wake modes, its coverage and communication capability are inevitably disrupted. Duty-cycling sensory devices directly leads to loss of sensing coverage, while duty-cycling radio transceivers directly leads to loss of network connectivity. It is therefore crucial to understand the performance degradation as a result of duty-cycling the sensor nodes, and to design good networking mechanisms that work well with low duty-cycled sensor networks.

In this report we aim at understanding the fundamental relationship between duty-cycling the radio transceivers and the resulting network connectivity. Specifically we will consider *random* duty-cycling where sensor nodes are on/awake with a certain probability (called the wake/active probability). The definition of connectivity refers to the existence of a route (consisting of active nodes) from each *active* node to every other *active* node in the network. While intuitively increasing nodes' transmission radius and decreasing nodes' active probability have opposite effects on the connectivity, it is less clear how they are related quantitatively to ensure connectivity. We will focus on understanding how these quantities scale as the network density increases, by studying the asymptotic connectivity of the network. Asymptotic connectivity in this context refers to the existence of a route (consisting of active nodes) from each *active* node to every other *active* node in the network, as the number of nodes approaches infinity.

More precisely, we consider the network with n nodes uniformly and independently placed in a unit square in $\Re^2$. Each node is awake with probability $p(n)$ and is connected to active neighbors within the range of transmission $R(n)$ when it is active. The problem under consideration is how $p(n)$ and $R(n)$ are related to ensure that the network is connected with high probability as n goes to infinity. An important prior work is [1]. Our network model is essentially the same as that studied in [1], with the only difference that in [1] the wake/active probability $p(n)$ is always 1. [1] showed that it is sufficient and necessary for each node to be connected to $\Theta(\log n)$ nearest neighbors to achieve asymptotic connectivity as n approaches infinity. Building on this result, in this study we show that the above randomly duty-cycled network is asymptotically connected with probability one if and only if the *average* number of active neighbors a node has is on the order of $\log(np(n))$. It has to be mentioned that this result cannot be obtained as a straightforward extension to [1] as discussed in more detail in subsequent sections.

Given the sufficient and necessary conditions for asymptotic connectivity, we further investigate the relationship between transmission radius and active probability with respect to energy consumption. We also study the quantitative relationship between transmission radius and sensing radius when both the asymptotic connectivity as well as certain coverage requirement need to be satisfied. Moreover, we discuss some important related issues further. Asymptotic connectivity in the sparse network is discussed while we considered it in the dense network in this paper. We redefine asymptotic connectivity to investigate how the sufficient and necessary conditions are affected. Some experiments on phase transitions of probability of connectivity are performed in the network with sleep/wake activity. At last, we evaluate transmission radius to show how it is affected by introducing active probability.

The rest of the paper is organized as follows. We present the network model and our main result in the next section, and discuss its relationship to the related work in Section III. In Section IV we give a number of preliminary results, and Section V outlines the proof of the main result. Sections VI, VII, VIII give more discussion of our main result in the context of energy, coverage, and some other issues. Section IX concludes the paper.

II. Network Model and Main Result

Consider a unit square in $\Re^2$, where n nodes are deployed uniformly and independently. Time is slotted. In each time slot, a node has a probability $p(n)$ of being awake or active, referred to as the *active probability*. An active node is connected to its active neighbors within a circle of radius $R(n)$, referred to as the transmission range. Such a network is said to be asymptotically connected if there exists a path of active nodes between any pair of two active nodes with high probability as the density n approaches infinity. In order to study the conditions under which such a network is asymptotically connected, we will utilize a number of results derived for a similar, but not duty-cycled network (i.e., where $p(n) = 1$ for all n). We begin by introducing the following types of networks/graphs that will be used in this report.

- $\mathcal{G}_p(n, R(n))$ denotes the duty-cycled network mentioned above, i.e., a network formed in a unit square where n nodes are deployed uniformly and independently. In this network a node is active with probability $p(n)$ and when active is connected to its active neighbors within a circle of radius $R(n)$.
- $\mathcal{G}(n, R(n))$ denotes a non-duty-cycled network formed in a unit square with n nodes deployed uniformly and independently. In this network a node is always active and is connected to neighbors within a circle of radius $R(n)$.
- $\mathcal{G}^\lambda(n, R(n))$ denotes a network formed as a Poisson point process with intensity n. In this network a node is always active and is connected to neighbors within a circle of radius $R(n)$.

- $\mathcal{F}(n, \phi_n)$ denotes a network formed in a unit square with n nodes deployed uniformly and independently. In this network a node is always active and is connected to its ϕ_n nearest neighbors.
- $\mathcal{F}^\lambda(n, \phi_n)$ denotes a network formed as a Poisson point process with intensity n. In this network a node is always active and is connected to its ϕ_n nearest neighbors.

The following notations are used throughout this paper. For two functions $f(n)$ and $g(n)$ defined on some subset of the real line, (1) $f(n) = O(g(n))$ implies that there exist numbers n_0 and M such that $|f(n)| \le M \cdot |g(n)|$ for all $n > n_0$ (asymptotic upper bound); (2) $f(n) = \Theta(g(n))$ implies that $f(n) = O(g(n))$ and $g(n) = O(f(n))$ (asymptotic tight bound); and (3) $f(n) = o(g(n))$ implies that $\lim_{n\to\infty} f(n)/g(n) = 0$ (asymptotically negligible).

Our main result is shown in the following theorem.

Theorem 1 *There exist two constants k_1 and k_2, $0 < k_1 < k_2$, such that:*

1) *for $np(n)R^2(n) = k_2 \log(np(n))$, we have*

$$\lim_{n\to\infty} Pr\{\mathcal{G}_p(n, R(n)) \text{ is connected }\} = 1 \ , \quad (1)$$

2) *for $np(n)R^2(n) = k_1 \log(np(n))$, we have*

$$\lim_{n\to\infty} Pr\{\mathcal{G}_p(n, R(n)) \text{ is disconnected }\} = 1 \ . \quad (2)$$

Eqn. (1) is also commonly viewed as a *sufficient* condition on connectivity and Eqn. (2) commonly viewed as a *necessary* condition on connectivity. Put together, $np(n)R^2(n) = \Theta(\log(np(n))$ can be viewed as the sufficient and necessary conditions for asymptotic connectivity. In subsequent sections we will also refer to these two equations as part I and part II of the theorem.

Below we sketch the idea of the proof of the above theorem and discuss this result within the context of other existing results on asymptotic connectivity.

Figure 1 summarizes the main idea of the proof, and illustrates where our technical contributions lie. The network we are interested in, $\mathcal{G}_p(n, R(n))$, is shown on the top left. To prove the theorem, we first show that if a Poisson network with intensity $np(n)$, i.e., $\mathcal{G}^\lambda(np(n), R(n))$, is asymptotically connected/disconnected given the condition $np(n)R(n)^2 = k\log(np(n))$ for some $k > 0$, then $\mathcal{G}_p(n, R(n))$ is asymptotically connected/disconnected given the same condition (for possibly different constants).

This process is illustrated by the arrow labeled with "A" in the figure. Conceptually, because of the random duty-cycling, there are only on average $np(n)$ nodes awake in the network at any instance of time. This makes the network $\mathcal{G}_p(n, R(n))$ behave like a Poisson network rather than one with a fixed number of nodes. However, in order to study asymptotic connectivity $np(n)$ needs to approach infinity, which renders inapplicable the standard result of approximating a binomial distribution with a Poisson distribution (which assumes a finite intensity). Although this seems a highly intuitive result, we were not able to find a prior proof. We give one such proof in Lemma 3, where we establish the Poisson approximation of a binomial distribution when $np(n) \to \infty$.

We next show that if the network $\mathcal{F}^\lambda(np(n), \phi_{np})$, i.e., a Poisson network with intensity $np(n)$ where each node is connected to its ϕ_{np} nearest neighbors, is asymptotically connected/disconnected given the condition $\phi_{np} = c\log(np(n))$, for some $c > 0$, then the network $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically connected/disconnected given the condition $np(n)R(n)^2 = k\log(np(n))$ for some $k > 0$.

This process is illustrated by the arrow labeled with "B" in the figure. Here $\mathcal{F}^\lambda(np(n), \phi_{np})$ is a Poisson network with ϕ_{np} neighbors for each node, and $\mathcal{G}^\lambda(np(n), R(n))$ is a Poisson network with neighbors within a finite radius $R(n)$ of each node. Note that for the latter, the condition $np(n)R(n)^2 = k\log(np(n))$ for some $k > 0$ is on the average number of neighbors a node has, whereas for the former the condition $\phi_{np} = c\log(np(n))$ for some $c > 0$ is on the actual number of neighbors a node has.

The last step is to show that network $\mathcal{F}^\lambda(np(n), \phi_{np})$ is asymptotically connected/disconnected given the condition $\phi_{np} = c\log(np(n))$, for some $c > 0$. This network is essentially the same as $\mathcal{F}^\lambda(n, \phi_n)$ (with a different intensity). This result is obtained in similar ways as in [2], which showed the same result for $\mathcal{F}(n, \phi_n)$. This step is illustrated by the arrow labeled with "C" in the figure.

III. Related Works

Two most relevant results to that studied in this paper are from [1] and [2], respectively. In particular, as mentioned above [1] studied a network of the type $\mathcal{F}(n, \phi_n)$, and it was shown that it is sufficient and necessary for each node to be connected to its

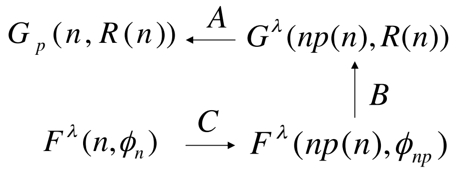


Fig. 1. Outline of the proof of Theorem 1.

$\Theta(\log n)$ nearest neighbors in order to achieve asymptotic connectivity for this network. An immediate thought was whether one could simply replace n with $np(n)$ in this result to obtain the conditions for a network of the type $\mathcal{G}_p(n, R(n))$, assuming $np(n) \to \infty$. Although intuitively appealing, there is a conceptual difference. Replacing n with $np(n)$ in this result implies that the sufficient and necessary conditions for asymptotic connectivity are for every active node to be connected to $np(n)$ nearest active neighbors. However, these conditions are not directly guaranteed when the neighborhood of each node is defined by a fixed radius $R(n)$ with randomly deployed nodes, and when the nodes are randomly duty-cycled. Instead, what Theorem 1 shows is that it is sufficient and necessary for each active node to be connected to an average of $\Theta(\log(np(n)))$ active neighbors for asymptotic connectivity of a network of the type $\mathcal{G}_p(n, R(n))$.

In [2] a network of the type $\mathcal{G}(n, R(n))$ was considered, and it was shown that with $\pi R^2(n) = \frac{\log n + c(n)}{n}$, the network is asymptotically connected with probability one if and only if $c(n) \to \infty$. This result is not directly used in our study. However, throughout this paper we follow heavily the basic definitions and methods used by [1] and [2], as well as use a number of (intermediate) results derived in them with appropriate modifications. These will be pointed out in subsequent sections.

[3] showed that the sufficient and necessary conditions for asymptotic coverage with connectivity in a *grid* network are $p(n)R^2(n) = \Theta(\frac{\log n}{n})$. Although mathematically similar, these conditions are not the same as the ones given by Theorem 1, since asymptotic coverage with connectivity is a different measure from asymptotic connectivity, and a grid network is different from a random network. [3] also showed that the sufficient condition for asymptotic connectivity in the grid network is in the form of

$$np(n)e^{-\frac{\pi p(n)R^2(n)n}{2}} \to 0 \text{ as } n \to \infty.$$

It can be shown that $p(n)R^2(n) = \Theta(\frac{\log n}{n})$ implies

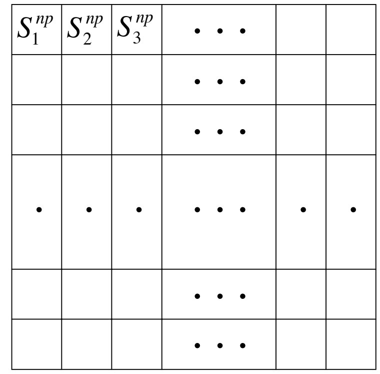


Fig. 2. The square tessellation τ_S^{np}.

$np(n)e^{-\frac{\pi p(n)R^2(n)n}{2}} \to 0$ as n and $np(n)$ both go to infinity. The reverse is not necessarily true. Therefore, we see that the condition for a randomly deployed network, i.e., $p(n)R^2(n) = \Theta(\frac{\log n}{n})$, is more restrictive than that for a grid network. Other related work includes [4], which studied the necessary and sufficient conditions of both asymptotic coverage and connectivity for a network with fixed node density λ but increasing area A. In addition, the concepts of k-connectivity and path connectivity were studied in [5] and [6], respectively.

IV. Preliminaries

For the proof of Theorem 1, we need the following definitions which were originally defined in [1], with slight generalization to account for $p(n) < 1$.

Definition 1 *Square tessellation* τ_S^{np}. The unit square is split equally into $M_{np} = \lceil \sqrt{\frac{np(n)}{K \log(np(n))}} \rceil^2$ small squares as depicted in Figure 2, where a constant $K > 0$ is a tunable parameter, and $\lceil x \rceil$ is the smallest integer larger than or equal to x. This tessellation of the unit square will be denoted by τ_S^{np}. The small squares are denoted by $S_i^{np}, i = 1, 2, \cdots, M_{np}$, from left to right, and from top to bottom.

Definition 2 *k-filling event.* Consider a structure composed of 21 squares each of side length $d/6$ and placed in a larger square of side length d: one at the center and the others at the periphery of the larger square with distance $d/4$ between the center square and the others. A k-filling event occurs if there are at least k nodes in each of 21 small squares and no nodes in the space between the center square and the others.

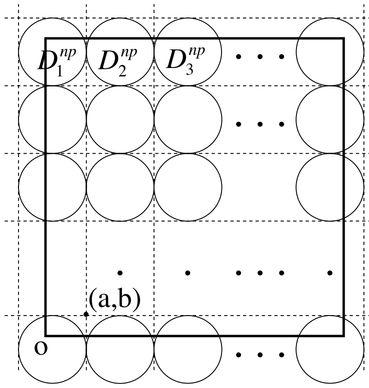


Fig. 3. The disk tessellation τ_D^{np}.

Definition 3 *Disk tessellation* $\tau_D^{np}(a,b)$. Consider a unit square with its bottom left corner being the origin, as shown in Figure 3. Let r be such that $\pi r^2 = \frac{K\log(np(n))}{np(n)}$, where $K > 0$ is a tunable parameter. Consider a grid of squares of size $2r$, with corners at $(a \mod 2r,\ b \mod 2r)$. Inside each square, we inscribe a disk of area $\frac{K\log(np(n))}{np(n)}$. The set of all disks intersecting the unit square are called the Disk Tessellation $\tau_D^{np}(a,b)$. The disks intersecting the unit square are denoted by $D_i^{np}, i = 1 \leq M_{np}$.

Throughout our analysis, the asymptotic regime of interest is where the duty cycle $p(n) \to 0$, $n \to \infty$ and $np(n) \to \infty$.

Consider the network $\mathcal{G}^\lambda(np(n), R(n))$, where $0 < p(n) < 1$. Denote the number of nodes that fall into the unit square by $\widetilde{M}_{np}$, and denote the number of nodes that fall into square S_i^{np} by $\widetilde{N}_i^{np}$.

Lemma 1 $\lim_{np(n)\to\infty} Pr\{|\widetilde{M}_{np} - np(n)| \leq \sqrt{np(n)\log(np(n))}\} = 1$.

Proof: Since $\widetilde{M}_{np}$ is a Poisson random variable with mean $= np(n)$ and variance $= np(n)$, by Chebychev's inequality,

$$\begin{aligned} Pr\{|\widetilde{M}_{np} - np(n)| > \sqrt{np(n)\log(np(n))}\} \\ \leq \frac{np(n)}{np(n)\log(np(n))} \\ = \frac{1}{\log(np(n))} \to 0, \text{ as } np(n) \to \infty. \end{aligned}$$

■

Consider $\mathcal{G}_p(n, R(n))$. Denote the number of active nodes in the unit square by M_n^a, which is a random variable. Denote the number of active nodes in square S_i^{np} by N_i^a.

Lemma 2 $\lim_{np(n)\to\infty} Pr\{|M_n^a - np(n)| \leq \sqrt{np(n)\log(np(n))}\} = 1$.

Proof: Since M_n^a is a Binomial random variable with mean $= np(n)$ and variance $= np(n)(1-p(n))$, by Chebychev inequality,

$$\begin{aligned} Pr\{|M_n^a - np(n)| > \sqrt{np(n)\log(np(n))}\}\} \\ \leq \frac{np(n)(1-p(n))}{np(n)\log(np(n))} \\ = \frac{1-p(n)}{\log(np(n))} \to 0, \text{ as } np(n) \to \infty. \end{aligned}$$

■

Let n be sufficiently large and p be small. When its product $np(n)$ is of moderate magnitude, the poisson approximation of binomial distribution is proven in the literature [7]. In this report, we need it in the special case of $np(n) \to \infty$ as $n \to \infty$, which will be proved in the following lemma.

Lemma 3 *Suppose that* $p(n) \to 0$ *and* $np(n) \to \infty$ *as* $n \to \infty$*. For any nonnegative* $j \leq n$ *and sufficiently large* n*,* $Pr\{M_n^a = j\}$ *is approximated by* $Pr\{\widetilde{M}_{np} = j\}$*, i.e., in the limit their difference goes to zero.*

Proof: We have that

$$Pr\{M_n^a = j\} = \binom{n}{j} p(n)^j (1-p(n))^{n-j} \ ,$$

$$Pr\{\widetilde{M}_{np} = j\} = \frac{(np(n))^j e^{-np(n)}}{j!}.$$

As $Pr\{M_n^a = j\}$ is a binomial distribution determined by n and $p(n)$, we will denote it by $b(j; n, p(n))$. Thus

$$b(0; n, p(n)) = (1-p(n))^n. \tag{3}$$

By the definition of the derivative of function $\log x$, we have

$$\lim_{\delta\to 0} \frac{\log x - \log(x-\delta)}{\delta} = \frac{1}{x} \ . \tag{4}$$

Since $p(n) \to 0$ as $n \to \infty$, Eqn. (4) can be written as

$$\lim_{n\to\infty} \frac{\log x - \log(x-p(n))}{p(n)} = \frac{1}{x}.$$

For $x = 1$, we have

$$\lim_{n\to\infty} \frac{-\log(1-p(n))}{p(n)} = 1.$$

In other words, $\forall \epsilon_1 > 0$, there exists $N_1 > 0$ such that $n > N_1$ implies $|\frac{-\log(1-p(n))}{p(n)} - 1| < \epsilon_1$. Let

$\Delta(n) \equiv \frac{-\log(1-p(n))}{p(n)} - 1$, such that $\Delta(n) \in [-\epsilon_1, \epsilon_1]$. For all $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists $N_2 > 0$ such that $n > \max\{N_1, N_2\}$ implies

$$\begin{aligned} & |(1-p(n))^n - e^{-np(n)}| \\ = & |(1-p(n))^{-\frac{1}{p(n)}\cdot(-np(n))} - e^{-np(n)}| \\ = & |e^{\frac{-1}{p(n)}\log(1-p(n))\cdot(-np(n))} - e^{-np(n)}| \\ = & |e^{(1+\Delta(n))\cdot(-np(n))} - e^{-np(n)}| \\ = & |e^{-np(n)}(e^{-np(n)\cdot\Delta(n)} - 1)|. \end{aligned} \quad (5)$$

Because $|\Delta(n)|$ is bounded by ϵ_1, $|e^{-np(n)\cdot\Delta(n)} - 1|$ is bounded by some $N_3 > 0$. Therefore, Eqn. (5) $\leq |e^{-np(n)}| \cdot N_3 < \epsilon_2$. Thus for sufficiently large n we have

$$b(0; n, p(n)) \approx e^{-np(n)}.$$

Furthermore, for any fixed j we have

$$\frac{b(j; n, p(n))}{b(j-1; n, p(n))} = \frac{np(n) - (j-1)p(n)}{j(1-p(n))}.$$

Therefore for sufficiently large n, we have

$$b(j; n, p(n)) \approx \frac{(np(n))^j}{j!} e^{-np(n)} = Pr\{\widetilde{M}_{np} = j\}.$$

■

Lemma 4 (Lemma 3.2.5 (iii) in [2]) *Suppose Y is a Poisson random variable with parameter λ, then for any $K > \frac{1}{\log(4/e)}$, we have*

$$\lim_{\lambda\to+\infty} \frac{1}{\lambda} e^{\frac{1}{K}\lambda} \cdot Pr\{|Y - \lambda| \geq \mu\lambda\} = 0, \forall \mu \in (\mu^*, 1),$$

where μ^* is the root of $-\mu^* + (1+\mu^*)\log(1+\mu^*) = \frac{1}{K}$.

Lemma 5 below is based on Lemma 3.1 in [2], with a slight modification by using $np(n)$ instead of n.

Lemma 5 *For any $K > \frac{1}{log(4/e)}$,*
$\lim_{np(n)\to\infty} Pr\{\max_i |\widetilde{N}_i^{np} - K\log(np(n))| \leq \mu K \log(np(n))\} = 1, \forall \mu \in (\mu^*, 1)$,
where $\mu^* \in (0,1)$ is the sole root of the equation $-\mu^* + (1+\mu^*)\log(1+\mu^*) = \frac{1}{K}$.

Proof: Recall $M_{np} \triangleq \frac{np(n)}{K\log(np(n))}$. By invoking the independence property of the Poisson process for the random variables $\widetilde{N}_1^{np}, \widetilde{N}_2^{np}, \cdots, \widetilde{N}^{np}_{\frac{np(n)}{K\log(np(n))}}$, we have

$$\begin{aligned} & Pr\{\max_{1\leq i\leq M_{np}} |\widetilde{N}_i^{np} - K\log(np(n))| \leq \mu K\log(np(n))\} \\ & = \prod_{i=1}^{M_{np}} Pr\{|\widetilde{N}_i^{np} - K\log(np(n))| \leq \mu K\log(np(n))\} \\ & = (Pr\{|\widetilde{N}_1^{np} - K\log(np(n))| \leq \mu K\log(np(n))\})^{M_{np}} \\ & = (1 - Pr\{|\widetilde{N}_1^{np} - K\log(np(n))| \\ & \qquad > \mu K\log(np(n))\})^{M_{np}} \\ & = e^{\frac{np(n)}{K\log(np(n))}\cdot\log(1 - Pr\{|\widetilde{N}_1^{np} - K\log(np(n))| > \mu K\log(np(n))\})}. \end{aligned}$$

If we let $\rho_{np} \triangleq K\log(np(n))$, which is the mean value of $\widetilde{N}_1^{np}$, then

$$\begin{aligned} & Pr\{\max_{1\leq i\leq M_{np}} |\widetilde{N}_i^{np} - K\log(np(n))| \leq \mu K\log(np(n))\} \\ & = \exp\{\frac{e^{\frac{\rho_{np}}{K}}}{\rho_{np}} \cdot \log(1 - Pr\{|\widetilde{N}_1^{np} - \rho_{np}| > \mu\rho_{np}\})\}. \end{aligned}$$

Since by Chebychev's inequality,

$$\begin{aligned} & Pr\{|\widetilde{N}_1^{np} - \rho_{np}| > \mu\rho_{np}\} \leq \frac{\text{var}(\widetilde{N}_1^{np})}{(\mu\rho_{np})^2} \\ & = \frac{\rho_{np}}{(\mu\rho_{np})^2} = \frac{1}{\mu^2\rho_{np}} \to 0, \text{ as } np(n) \to \infty, \end{aligned}$$

and $\log(1-x)$ is approximated by $-x$ for small x, we have

$$\begin{aligned} & Pr\{\max_{1\leq i\leq M_{np}} |\widetilde{N}_i^{np} - K\log(np(n))| \leq \mu K\log(np(n))\} \\ & = e^{-\frac{e^{\frac{\rho_{np}}{K}}}{\rho_{np}}\cdot Pr\{|\widetilde{N}_1^{np} - \rho_{np}| > \mu\rho_{np}\}\cdot(1+o(1))}. \end{aligned}$$

Hence, by Lemma 4, we deduce that

$$\begin{aligned} & Pr\{\max_{1\leq i\leq M_{np}} |\widetilde{N}_i^{np} - K\log(np(n))| \leq \mu K\log(np(n))\} \\ & \to 1, \text{ as } np(n) \to \infty. \end{aligned}$$

■

Consider the disk tessellation $\tau_D^{np}(a,b)$ in a unit square with nodes deployed as a Poisson point process with intensity $np(n)$. Similarly to the square tessellation, let the number of nodes that fall into disk D_i^{np} be denoted as $\widetilde{N}_{D,i}^{np}$.

Lemma 6 *For any $K > \frac{1}{log(4/e)}$ and any point sequence $\{(a_n, b_n)\} \in \mathbb{R}^2, n = 1, 2, \cdots\}$,*
$\lim_{np(n)\to\infty} Pr\{\widetilde{N}_{D,i}^{np} \leq (1+\mu)K\log(np(n))$, *for any disk D_i^{np} in tessellation $\tau_D^{np}(a_n, b_n)\} = 1, \forall \mu \in (\mu^*, 1)$,*

where $\mu^* \in (0,1)$ is the sole root of the equation $-\mu^* + (1+\mu^*)\log(1+\mu^*) = \frac{1}{K}$.

Let us consider graph $\mathcal{G}^\lambda(np(n), R(n))$. Let $P^{\lambda;(1)}(np(n), R(n))$ be the probability that $\mathcal{G}^\lambda(np(n), R(n))$ has at least one isolated node (i.e, one with no neighbors) and $P_d^\lambda(np(n), R(n))$ be the probability that $\mathcal{G}^\lambda(np(n), R(n))$ is disconnected. From *continuum percolation* [8], we know that $P_d^\lambda(np(n), R(n))$ is asymptotically the same as $P^{\lambda;(1)}(np(n), R(n))$. Consider $\mathcal{G}(np(n), R(n))$, the network with exactly $np(n)$ number of nodes. Let $P_d(np(n), R(n))$ be the probability that $\mathcal{G}(np(n), R(n))$ is disconnected.

Lemma 7 (Lemma 3.1 in [2]) *If $\pi R^2(n) = \frac{\log(np(n))+c(n)}{np(n)}$, then*

$$\limsup_{np(n)\to\infty} P^{\lambda;(1)}(np(n), R(n)) \le e^{-c},$$

where $c = \lim_{n\to\infty} c(n)$.

Lemma 8 (Theorem 2.1 in [2]) *If $\pi R^2(n) = \frac{\log(np(n))+c(n)}{np(n)}$, then*

$$\liminf_{np(n)\to\infty} P_d(np(n), R(n)) \ge e^{-c}(1-e^{-c}),$$

where $c = \lim_{n\to\infty} c(n)$.

The following theorem is proven using intermediate results in [2].

Theorem 2 *The network $\mathcal{G}^\lambda(np(n), R(n))$ with $\pi R^2(n) = \frac{\log(np(n))+c(n)}{np(n)}$ is connected with probability one as $np(n) \to \infty$ and $n \to \infty$ if and only if $\lim_{n\to\infty} c(n) = \infty$.*

Proof: (Sufficiency) From percolation theory, for any $\epsilon > 0$ and for all sufficiently large $np(n)$, we have

$$\begin{aligned} P_d^\lambda(np(n), R(n)) &\le (1+\epsilon)P^{\lambda;(1)}(np(n), R(n)) \\ &\le (1+\epsilon)e^{-c(n)} \ , \end{aligned}$$

where the second inequality is from Lemma 7. Since $\epsilon > 0$ is arbitrary, we have

$$\limsup_{np(n)\to\infty} P_d^\lambda(np(n), R(n)) \le e^{-c}.$$

(Necessity) From Eqn. (1.21) in [2],

$$\begin{aligned} & P_d^\lambda(np(n), R(n)) \\ \ge\ & P_d(np(n), R(n))(\frac{1}{2}-\epsilon) - \frac{e^{-np(n)\pi R^2(n)}}{\pi R^2(n)} \\ \ge\ & P_d(np(n), R(n))(\frac{1}{2}-\epsilon) - \frac{e^{-c(n)}}{\log(np(n))+c(n)}. \end{aligned}$$

Based on Lemma 8, since $\epsilon > 0$ is arbitrary,

$$\liminf_{np(n)\to\infty} P_d^\lambda(np(n), R(n)) \ge \frac{1}{2}e^{-c}(1-e^{-c}).$$

■

V. Proof of Theorem 1

In this section, we prove both two parts of Theorem 1. For simplicity we will ignore edge effect in our discussion, but note that edge effect does not alter the main theorem (see also [1], [2]). The proof of each part consists of three steps. In part I, the proof proceeds as follows:

(1) Given $np(n)R(n)^2 = k_2 \log(np(n))$ for some $k_2 > 0$, we show $\mathcal{G}_p(n, R(n))$ is asymptotically connected if $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically connected.

(2) It is shown that if there exists c_2 such that $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ is asymptotically connected, then there exists k_2 such that $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically connected with $np(n)R(n)^2 = k_2\log(np(n))$.

(3) We show that $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ is asymptotically connected for some $c_2 > 0$.

For the first step, note that $R(n)$ is bounded and that $n \to \infty$ implies $np(n) \to \infty$. For sufficiently large n,

$$\begin{aligned}
&Pr\{\mathcal{G}_p(n,R(n)) \text{ is connected}\}\\
&=\sum_{j=0}^{n} Pr\{\mathcal{G}_p(n,R(n)) \text{ is connected}|M_n^a=j\}\\
&\quad\cdot Pr\{M_n^a=j\}\\
&=(\sum_{|j-np(n)|\le\sqrt{(np(n)\log(np(n)))}} + \sum_{otherwise})\\
&\quad Pr\{\mathcal{G}_p(n,R(n)) \text{ is connected}|M_n^a=j\}\\
&\quad\cdot Pr\{M_n^a=j\}\\
&=\sum_{|j-np(n)|\le\sqrt{(np(n)\log(np(n)))}} Pr\{\mathcal{G}_p(n,R(n)) \text{ is}\\
&\quad \text{connected}|M_n^a=j\}\cdot Pr\{M_n^a=j\}+o(1)\\
&=\sum_{|j-np(n)|\le\sqrt{(np(n)\log(np(n)))}} Pr\{\mathcal{G}^\lambda(np(n),R(n))\\
&\quad \text{is connected}|\widetilde{M}_{np}=j\}\cdot Pr\{\widetilde{M}_{np}=j\}\\
&\quad\cdot(1+o(1))+o(1)\ ,
\end{aligned} \tag{6}$$

where the third equality is based on Lemma 1. The fourth equality is based on Lemma 3 and the fact that $\mathcal{G}_p(n,R(n))$ given j active nodes is the same as $\mathcal{G}^\lambda(np(n),R(n))$ given j nodes are in the network. From Lemma 2 we have that Eqn. (6) can be written as

$$\begin{aligned}
&(1+o(1))\cdot(Pr\{\mathcal{G}^\lambda(np(n),R(n)) \text{ is connected}\}\\
&\quad+o(1))+o(1).
\end{aligned}$$

Therefore if

$$\lim_{n\to\infty} Pr\{\mathcal{G}^\lambda(np(n),R(n)) \text{ is connected}\}=1,$$

then

$$\lim_{n\to\infty} Pr\{\mathcal{G}_p(n,R(n)) \text{ is connected}\}=1\ ,$$

thus completing the first step.

In Step 2 we show that if there exists c_2 for $\mathcal{F}^\lambda(np(n),c_2\log(np(n)))$ to be asymptotically connected, then there exists k_2 for $\mathcal{G}^\lambda(np(n),R(n))$ to be asymptotically connected with $np(n)R(n)^2 = k_2\log(np(n))$. To prove this, let us tessellate $\mathcal{G}^\lambda(np(n),R(n))$ by τ_S^{np}, with K, μ satisfying Lemma 5. Consider some nodes whose radius is $R(n) = \sqrt{\frac{2K\log(np(n))}{np(n)}}$ on τ_S^{np}, as shown in Figure 4. Every circle contains a small square. From Lemma 5, we know that each circle contains more than or equal to $K(1-\mu)\log(np(n))$ nodes with high probability, where $\mu\in(\mu^*,1)$. We construct another graph by connecting each node with its nearest $K(1-\mu)\log(np(n))-1$ neighbors, which is $\mathcal{F}^\lambda(np(n),K(1-\mu)\log(np(n))-1)$. If $\mathcal{F}^\lambda(np(n),K(1-\mu)\log(np(n))-1)$ is asymptotically connected, then $\mathcal{G}^\lambda(np(n),R(n))$ with $np(n)R(n)^2=2K\log(np(n))$ is asymptotically connected. Thus there exists $k_2=2K$ when $c_2=K(1-\mu)$. This completes the second step.

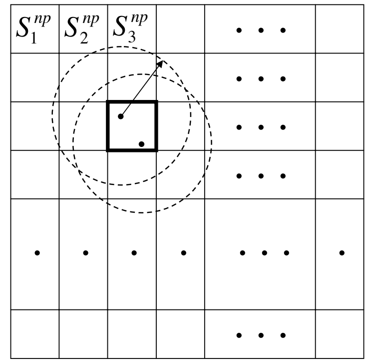


Fig. 4. Nodes with radius of transmission $R(n) = \sqrt{\frac{2K\log(np(n))}{np(n)}}$ on τ_S^{np}.

Finally, we want to prove that $\mathcal{F}^\lambda(np(n),c_2\log(np(n)))$ is asymptotically connected for $c_2>\frac{2}{\log(4/e)}$. It suffices to show that for some $\delta>0$,

$$\lim_{n\to\infty} Pr\{\mathcal{F}^\lambda(np(n),(2/\log(4/e)+\delta)\log(np(n))) \text{ is connected}\}=1.$$

According to Lemma 6, $\mu^*\to 1$ as $K\to(1/\log(4/e))^+$. So for any $\delta>0$, there is a constant $\delta'>0$ such that

$$\begin{aligned}
K=&1/\log(4/e)+\delta'\\
&\Rightarrow(1+\mu^*)K<2/\log(4/e)+\delta.
\end{aligned}\tag{7}$$

For the rest of this proof, we fix the parameter K in the Disk tessellation to be the one in Eqn. (7), and fix μ such that

$$1>\mu>\mu^* \text{ and } (1+\mu)K<2/\log(4/e)+\delta.$$

Let $r_{np}\triangleq\sqrt{\frac{K\log(np(n))}{\pi np(n)}}$ be the radius of the disks in the Disk tessellation. Then choose two positive constants ϵ,$\eta\in(0,1)$ such that

$$\pi(r_{np}-\epsilon r_{np})^2>\frac{(1+\eta)\log(np(n))}{np(n)}. \tag{8}$$

Now let us tessellate the unit square by a collection of several disk tessellations:

$$\tau_\epsilon^{np} \triangleq \{\tau_D^{np}(i\cdot\epsilon r_{np}, j\cdot\epsilon r_{np}), i, j = 0, 1, 2, \cdots, 2\cdot[\frac{1}{\epsilon}]+1\}.$$

This collection of tessellations has the following property: For any point (a, b) in the unit square, there is a disk in τ_ϵ^{np} whose center is within a distance of ϵr_{np} from the point (see Figure 3). Since the number of tessellations in τ_ϵ^{np} is finite, by Lemma 6, we know that

$$Pr\{\text{Every disk of } \tau_\epsilon^{np} \text{ contains no more than } (2/\log(4/e)+\delta)\log(np(n)) \text{ nodes}\} \to 1, \text{ as } n\to\infty.$$

By the choice of r_{np}, ϵ and τ_ϵ^{np}, any disk with radius $(1-\epsilon)r_{np}$ and centered in the unit square will be contained in a disk in the collection of tessellations τ_ϵ^{np} (see Figure 4). So if any of the disks of the tessellation collection τ_ϵ^{np} contains no more than $(2/\log(4/e)+\delta)\log(np(n))$ nodes, then each node of $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ will be connected to every node that is within distance of $(1-\epsilon)r_{np}$. So if we define $B_{np} \triangleq \{\text{Every disk of } \tau_\epsilon^{np} \text{ contains no more than } (\frac{2}{\log(4/e)}+\delta)\log(np(n)) \text{ nodes}\}$, then

$$Pr\{\mathcal{F}^\lambda(np(n), (2/\log(4/e)+\delta)\log(np(n))) \text{ is connected}|B_{np}\} \to 1, \text{ as } n\to\infty.$$

Therefore

$$\begin{aligned} &Pr\{\mathcal{F}^\lambda(np(n), (2/\log(4/e)+\delta)\log(np(n))) \text{ is connected}\} \\ &= Pr\{B_{np}\}\cdot Pr\{\mathcal{F}^\lambda(np(n), (2/\log(4/e)+\delta) \log(np(n))) \text{ is connected}|B_{np}\} \\ &\quad + Pr\{B_{np}^c\}\cdot Pr\{\mathcal{F}^\lambda(np(n), (2/\log(4/e)+\delta) \log(np(n))) \text{ is connected}|B_{np}^c\} \\ &= (1+o(1))\cdot(1+o(1))+o(1) \to 1, \text{ as } n\to\infty. \end{aligned}$$

Hence we proved that $\mathcal{F}^\lambda(np(n), c_2\log(np(n)))$ is asymptotically connected for $c_2 > \frac{2}{\log(4/e)}$, completing the third step.

The proof of the second part of Theorem 1 follows a very similar procedure, consisting of three steps:

(1) Given $np(n)R(n)^2 = k_1\log(np(n))$ for some $k_1 > 0$, we show $\mathcal{G}_p(n, R(n))$ is asymptotically disconnected if $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically disconnected.

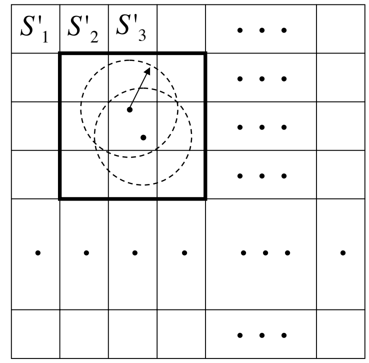


Fig. 5. Nodes with radius of transmission $R(n) = \sqrt{\frac{K'\log(np(n))}{np(n)}}$ on $\tau_{S'}^{np}$.

(2) It is shown that if there exists c_1 such that $\mathcal{F}^\lambda(np(n), c_1\log(np(n)))$ is asymptotically disconnected, then there exists k_1 such that $\mathcal{G}^\lambda(np(n), R(n))$ is asymptotically disconnected with $np(n)R(n)^2 = k_1\log(np(n))$.

(3) We show that $\mathcal{F}^\lambda(np(n), c_1\log(np(n)))$ is asymptotically disconnected for some $c_1 > 0$.

In the first step, similar to part I we will use the fact that $n\to\infty$ implies $np(n)\to\infty$. With slight modification from connectivity to disconnectivity on the argument used in part I of the proof given early, one can easily show that if $\lim_{n\to\infty} Pr\{\mathcal{G}^\lambda(np(n), R(n)) \text{ is disconnected}\} = 1$, then $\lim_{n\to\infty} Pr\{\mathcal{G}_p(n, R(n)) \text{ is disconnected}\} = 1$. This completes the first step of the proof of part II.

In the second step we show that if there exists c_1 such that $\mathcal{F}^\lambda(np(n), c_1\log(np(n)))$ is asymptotically disconnected, then there exists k_1 such that $\mathcal{G}^\lambda(np(n), R(n))$ with $np(n)R(n)^2 = k_1\log(np(n))$ is asymptotically disconnected. To prove this, we tessellate $\mathcal{G}^\lambda(np(n), R(n))$ by τ_S^{np}, with K, μ satisfying Lemma 5. Furthermore, we split each square into $\lceil\sqrt{\frac{9\cdot 21(1+\mu)}{1-\mu}}\rceil^2$ smaller squares. Denote by $\tau_{S'}^{np}$ the new tessellation with $\lceil\sqrt{\frac{np(n)}{K\log(np(n))}}\rceil^2 \cdot \lceil\sqrt{\frac{9\cdot 21(1+\mu)}{1-\mu}}\rceil^2$ squares and let $\widetilde{N}_i^*$ be the number of nodes in each smaller square S_i'. Thus $\widetilde{N}_i^*$ is a Poisson random variable with mean $\frac{K(1-\mu)}{9\cdot 21(1+\mu)}\log(np(n))$. Similarly to Lemma 5, for $K' > \frac{1}{\log(4/e)}$, we have

$$\lim_{n\to\infty} Pr\{\max_i \widetilde{N}_i^* \le (1+\mu)K'\log(np(n))\} = 1, \quad \forall\mu\in(\mu^{**}, 1), \tag{9}$$

where $K' = \frac{1-\mu}{9 \cdot 21(1+\mu)}K$ and μ^{**} is the root of $-\mu^{**} + (1+\mu^{**})\log(1+\mu^{**}) = \frac{1}{K'}$.

Consider some nodes with radius $R(n) = \sqrt{\frac{K' \log(np(n))}{np(n)}}$, the side length of each smaller square on $\tau_{S'}^{np}$ as shown in Figure 5. Every circle is included in a group of at most 9 small squares. From Eqn. (9), each circle contains less than or equal to $\frac{K(1-\mu)}{21}\log(np(n))$ nodes with high probability. We can thus construct another graph by connecting each node with its nearest $\frac{K(1-\mu)}{21}\log(np(n)) - 1$ neighbors, which results in $\mathcal{F}^{\lambda}(np(n), \frac{K(1-\mu)}{21}\log(np(n)) - 1)$. Consequently, if $\mathcal{F}^{\lambda}(np(n), \frac{K(1-\mu)}{21}\log(np(n)) - 1)$ is asymptotically disconnected, $\mathcal{G}^{\lambda}(np(n), R(n))$ with $np(n)R(n)^2 = \frac{1-\mu}{9 \cdot 21(1+\mu)}K\log(np(n))$ is asymptotically disconnected. Note that for large $np(n)$, $\frac{K(1-\mu)}{21}\log(np(n)) \gg 1$. Thus there exists $k_1 = \frac{1-\mu}{9 \cdot 21(1+\mu)}K$ when $c_2 = \frac{K(1-\mu)}{21}$. This completes the second step of the proof.

Finally, we want to prove that $\mathcal{F}^{\lambda}(np(n), c_1\log(np(n)))$ is asymptotically disconnected for $c_1 < \frac{(1-\mu)K}{21}$. It suffices to show that for some $\epsilon > 0$,

$$\lim_{n\to\infty} Pr\{\mathcal{F}^{\lambda}(np(n), \epsilon\log(np(n)))) \text{ is connected}\} = 0.$$

According to Lemma 5,

$$\lim_{n\to\infty} Pr\{\max_i |\widetilde{N}_i^{np} - K\log(np(n))| \le \mu K\log(np(n))\} = 1.$$

Therefore if we let

$$A_i^{np} \triangleq \{\text{ No } (\epsilon\log(np(n)) + 1)\text{-filling event occurs in the trap of } S_i^{np}\},$$
$$Q^{np} \triangleq \{(k_1, k_2, \cdots, k_{M_{np}}) : k_1 + k_2 + \cdots + k_{M_{np}} = \widetilde{M}^{np}, \text{ where } \widetilde{M}^{np} \ge 0 \text{ and } k_i \ge 0, \forall i\},$$

then we have

$$\begin{aligned} &Pr\{\mathcal{F}^{\lambda}(np(n), \epsilon\log(np(n)))) \text{ is connected}\} \\ &\le Pr\{A_i^{np}, \forall i\} \\ &= \Sigma_{(k_1,k_2,\cdots,k_{M^{np}})\in Q^{np}} Pr\{A_i^{np}, \forall i; \widetilde{N}_i^{np} = k_i, \forall i\} \\ &= \Sigma_{(k_1,k_2,\cdots,k_{M^{np}})\in Q^{np}} Pr\{A_i^{np}, \forall i | \widetilde{N}_i^{np} = k_i, \forall i\} \\ &\qquad \cdot Pr\{\widetilde{N}_i^{np} = k_i, \forall i\} \end{aligned}$$

The last step of this proof is the same as the proof of the necessity part in [1], replacing n with $np(n)$.

VI. Energy Efficiency with Asymptotic Connectivity

Low duty-cycling was originally proposed to conserve idling energy as described in Section I. That is, total energy consumption may be reduced with smaller $p(n)$ with and larger $R(n)$. Theorem 1 gives us the idea how $R(n)$ and $p(n)$ scale as n grows in order to maintain connectivity between nodes in the network. Given the condition for asymptotic connectivity, it is interesting to show that under what conditions of $R(n)$ and $p(n)$ minimize total energy consumption of nodes.

We adopt the energy model from [9]. Energy spent in transmission is a function of $R(n)$ and the number of bits to transmit. Energy spent in reception and one spent in sensing are only functions of the number of bits. Let E denote the average energy consumed by a node for a unit time, which is

$$E = (C_1 R(n)^{\alpha} + C_2)p(n),$$

where α is a constant which depends on the attenuation of the signal in the environment. We will use the common values of $\alpha = 2$ and $\alpha = 4$. C_1 and C_2 are constants which depends on the number of bits, the energy used by the circuitry for every bit, and so on.

We want to minimize E with respect to the condition from Theorem 1. For some $k > 0$,

$$\min_{R(n),p(n)} (C_1 R(n)^{\alpha} + C_2)p(n)$$
$$\text{w.r.t. } p(n)R^2(n) = k\frac{\log(np(n))}{n}. \tag{10}$$

- Suppose $\alpha = 2$. Optimizing Eqn. (10) is equivalent to show the following equation.

$$\min_{p(n)} \frac{C_1 k\log(np(n))}{n} + C_2 p(n). \tag{11}$$

  Eqn. (11) is an increasing function of $p(n)$. That is, smaller $p(n)$ results in smaller energy consumption. The more sensors in sleep, the less energy consumed. Then, how many sensors to put into sleep depends on constraint on delay and data rate that each node carries. This will be related to the number of hops to traverse, which is also a function of $R(n)$. This can be studied as future research.
- When $\alpha = 4$, Eqn. (10) becomes

$$\min_{p(n)} \frac{C_1 k^2 \log^2(np(n))}{n^2 p(n)} + C_2 p(n). \tag{12}$$

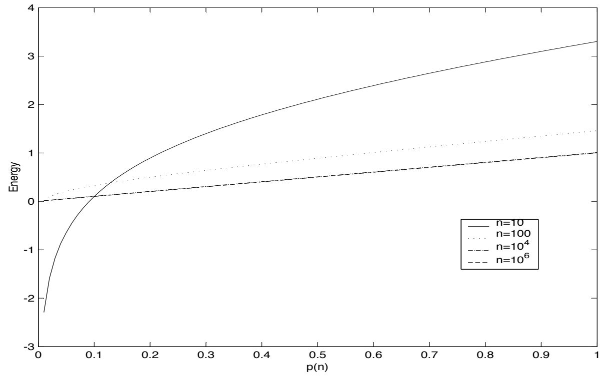


Fig. 6. Energy consumption when $C_1 = 1$,$C_2 = 1$,$k = 10$ and $\alpha = 2$.

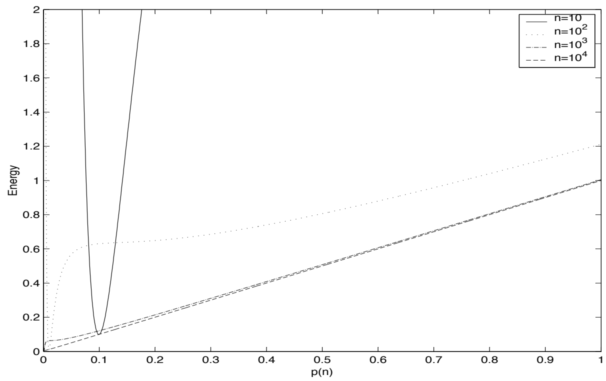


Fig. 7. Energy consumption when $C_1 = 1$,$C_2 = 1$,$k = 10$ and $\alpha = 4$.

Eqn. (12) is minimized when $p(n) = \frac{1}{n}e^{2/n}$. In order to maintain asymptotic connectivity, $p(n)$ cannot decrease faster than $\frac{1}{n}$. While asymptotic connectivity is guaranteed, energy consumption is minimized if $p(n)$ decreases to zero with the rate $\frac{1}{n}e^{2/n}$ as n increases.

We evaluate Eqn. (10) to give numerical results when $\alpha = 2$ and $\alpha = 4$. Figure 6 shows that energy consumption is an increasing function of $p(n)$ as in Eqn. (11) when $\alpha = 2$. When $\alpha = 4$, there exists a minimum of energy consumption at $p(n) = \frac{1}{n}e^{2/n}$ as shown in Figure 7. When n becomes large, $p(n)$ approaches to zero. Thus, energy consumption is approximately linearly proportional to $p(n)$ for large n as ones in Figure 6.

VII. Asymptotic Connectivity with Coverage

Suppose that *inactive* nodes turn off both the radio transceiver and sensory device. And let $r(n)$ be the sensing radius. We are interested in a condition for asymptotic connectivity while we give a certain level of coverage.

We consider $\mathcal{G}_p(n, R(n))$. Given that the long term average sleep ratio of a node is $q(n) := 1 - p(n)$, regardless of the distribution of the on and off periods (assuming they are both of finite mean which is desirable for coverage purpose), the probability that a given point in a unit square is not covered by any active node is

$$P_u = \sum_{j=0}^{n} q(n)^j \binom{n}{j} (\pi r^2(n))^j (1 - \pi r^2(n))^{n-j} .$$

where $n\pi r^2(n)$ is the expected number of nodes deployed within a circle of radius $r(n)$ around the point. The associated joint probability of the point being uncovered and at least one node being within a circle of radius $r(n)$ is

$$P_{u,c} = \sum_{j=1}^{n} q(n)^j \binom{n}{j} (\pi r^2(n))^j (1 - \pi r^2(n))^{n-j} .$$

$$\begin{aligned}
\lim_{n\to\infty} P_{u,c} &= \lim_{n\to\infty} \sum_{j=1}^{n} q(n)^j \frac{(n\pi r^2(n))^j e^{-n\pi r^2(n)}}{j!} \\
&= \lim_{n\to\infty} e^{-n\pi r^2(n)(1-q(n))}(1 - e^{-n\pi r^2(n)q(n)}) \\
&\quad - \lim_{n\to\infty} e^{-n\pi r^2(n)(1-q(n))} \\
&\quad \cdot \sum_{j=n+1}^{\infty} \frac{(n\pi r^2(n)q(n))^j e^{-n\pi r^2(n)q(n)}}{j!} \\
&= \lim_{n\to\infty} e^{-n\pi r^2(n)(1-q(n))}(1 - e^{-n\pi r^2(n)q(n)}).
\end{aligned}$$

First equality is from Lemma 3. The last equality is because $\sum_{j=n+1}^{\infty} \frac{(n\pi r^2(n)q(n))^j e^{-n\pi r^2(n)q(n)}}{j!}$ converges to 0 as n goes to infinity. Furthermore, we can obtain $q(n) = \frac{\log(V + e^{-n\pi r^2(n)})}{n\pi r^2(n)} + 1$ for fixed $P_{u,c} = V$ when n goes to infinity.

To achieve asymptotic connectivity for active nodes, $np(n)\pi R^2(n) = k_2 \log n$ from Theorem 1. This relationship gives us a boundary condition for asymptotic connectivity. Furthermore, a random sleep schedule is $p(n) = -\frac{\log(V + e^{-n\pi r^2(n)})}{n\pi r^2(n)}$ to achieve

$P_{u,c} = V$. Therefore, to achieve both,

$$n[-\frac{\log(V + e^{-n\pi r^2(n)})}{n\pi r^2(n)}]\pi R^2(n)$$
$$= k_2 \log[-\frac{n\log(V + e^{-n\pi r^2(n)})}{n\pi r^2(n)}]$$
$$\Rightarrow -\frac{R^2(n)}{r^2(n)} \log(V + e^{-n\pi r^2(n)})$$
$$= k_2 \log[-\frac{1}{\pi r^2(n)} \log(V + e^{-n\pi r^2(n)})]. \quad (13)$$

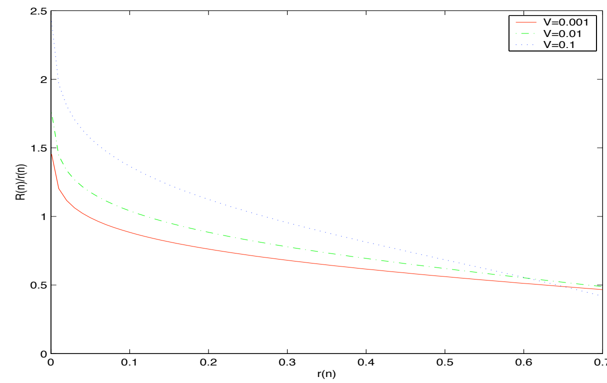
First, for fixed n and when V is very small, Eqn. (13) becomes

$$-\frac{R^2(n)}{r^2(n)} \log e^{-n\pi r^2(n)}$$
$$= k_2 \log(-\frac{1}{\pi r^2(n)} \log e^{-n\pi r^2(n)})$$
$$\Rightarrow n\pi R^2(n) = k_2 \log n.$$

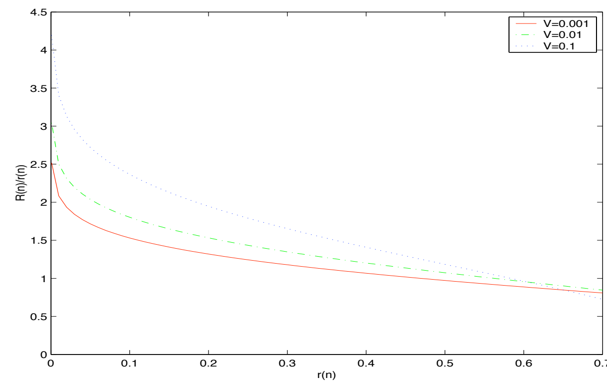
Second, for fixed V and when n is very large, Eqn. (13) becomes

$$-\frac{R^2(n)}{r^2(n)} \log V = k_2 \log(-\frac{1}{\pi r^2(n)} \log V)$$
$$\Rightarrow \frac{R^2(n)}{r^2(n)} \log \frac{1}{V} + k_2 \log r^2(n)$$
$$= k_2(\log\log \frac{1}{V} - \log \pi) \quad (14)$$

We evaluate Eqn. (14) to give numerical results in Figure 8. It shows boundary conditions for $\frac{R(n)}{r(n)}$ for asymptotic connectivity with some levels of coverage V. Each graph gives a ratio of $R(n)$ to $r(n)$ as $r(n)$ increases for $k_2 = 1$ and $k_2 = 3$. As $r(n)$ increases, $R(n)$ increases at a slower rate than $r(n)$ given V and k_2. As V is larger, $r(n)$ is smaller and the ratio gets larger. If k_2 is larger, it increase $R(n)$ or $p(n)$. Thus, the ratio of $R(n)$ over $r(n)$ gets bigger.



(a) $k_2 = 1$



(b) $k_2 = 3$

Fig. 8. Boundary conditions for $\frac{R(n)}{r(n)}$ for p-asymptotic connectivity with some levels of coverage V when $k_2 = 1$ and 3.

VIII. Discussion

A. Sparse Network

[10] studied the scaling property of connectivity for sparse networks where the node density d is fixed and the number of nodes n grows to infinity as well as the area A. Authors showed that the sufficient condition for the network to be connected is $r^2 n = kA \log \sqrt{A}$ for some constant $k > 0$. If we put dA instead n, the condition becomes $r^2 d = k \log \sqrt{A}$. This result can be viewed as average $\Theta(\log A) = \Theta(\log(n/d)) = \Theta(\log n)$ nearest neighbors are sufficient for the network to be connected.

Consider results in [1], which considered dense networks. Authors assumed fixed unit area and therefore transmission radius r is bounded. To investigate the conditions for asymptotic connectivity of the network, the number of nodes n grows to infinity as well as the node density d. They showed that the network is asymptotically connected if each node has $\Theta(\log n)$ nearest neighbors, which leads to the same result as in [10]. Although the results in [1] are proven for any fixed area A, we conclude that the result for dense networks can be applied to sparse networks.

In this paper, we considered the network formed in the unit square. As we argued above for results in [1], the result in this paper can be applied directly to sparse networks.

B. Phase Transition

[2] showed that probability of connectivity has zero-one transitions for large n in the *random* network with fixed radius $R(n)$. In this paper, we showed it for large n in the *random* network with fixed radius

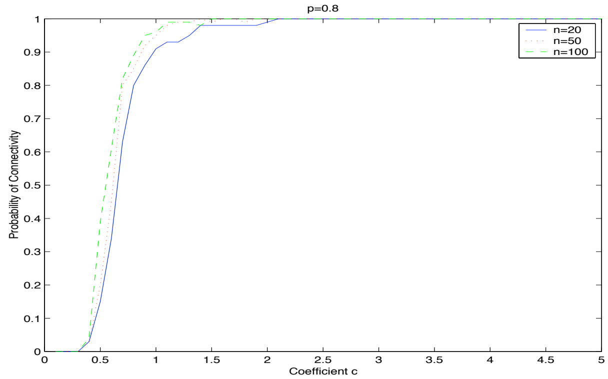


Fig. 9. Phase transition in probability of connectivity when $p(n)R^2(n) = c\frac{\log n}{n}$ and $p(n) = 0.8$.

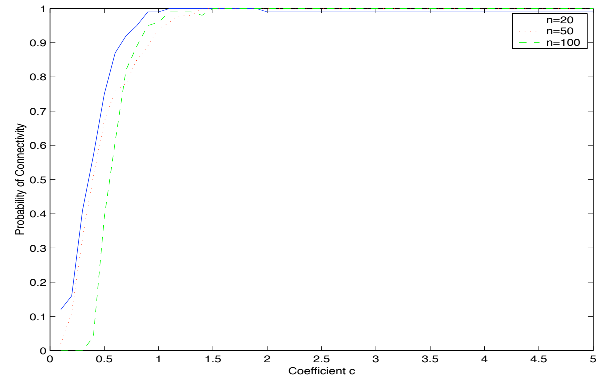


Fig. 10. Phase transition in probability of connectivity when $p(n)R^2(n) = c\frac{\log n}{n}$ and $p(n) = \frac{1}{\sqrt{n}}$.

$R(n)$ and active probability $p(n)$. We experiment zero-one transitions in probability of connectivity in the *random* network when $p(n)R^2(n) = \Theta(\frac{\log n}{n})$ and $R(n)$ is bounded. Note that the condition of bounded $R(n)$ is implied in the network formulation considered in this paper.

Figure 9 depicts probability of connectivity when $p(n)R^2(n) = c\frac{\log n}{n}$ with respect to c for $n = 20, 50, 100$ and fixed $p(n) = 0.8$. When n gets larger, the transitions become shaper. Figure 10 depicts probability of connectivity when $p(n)R^2(n) = c\frac{\log n}{n}$ with respect to c for $n = 20, 50, 100$ and $p(n) = \frac{1}{\sqrt{n}}$. In this case, $R(n)$ decreases to zero as n goes to infinity. Therefore, $np(n)$ goes to infinity and $R(n)$ is bounded. On the other hand, if $p(n)$ decreases faster than or equal to $1/n$, $np(n)$ does not go to infinity and $R(n)$ becomes unbounded. Therefore, such cases are unable to be satisfied in this network. This is illustrated in Figure 11, which depicts probability of connectivity when $p(n)R^2(n) = c\frac{\log n}{n}$ with respect to c for $n = 20, 50, 100$ and $p(n) = \frac{1}{n}$. As c increases, probability of connectivity never reaches to one.

C. *Radius with Low Duty-Cycling*

We showed in Theorem 1 that the network $\mathcal{G}_p(n, R(n))$ is asymptotically connected given the condition $nR^2(n)p(n) = K\log(np(n))$ for some $K > 0$. It is of interest to quantitatively measure the increase of transmission radius when nodes are low duty-cycled in order to maintain asymptotic connectivity. Intuitively, transmission radius with duty-cycling becomes larger than one with no duty-cycling, i.e., $p(n) = 1$. Let us denote the transmission radius with duty-cycling by $R(n)$ and one with no duty-

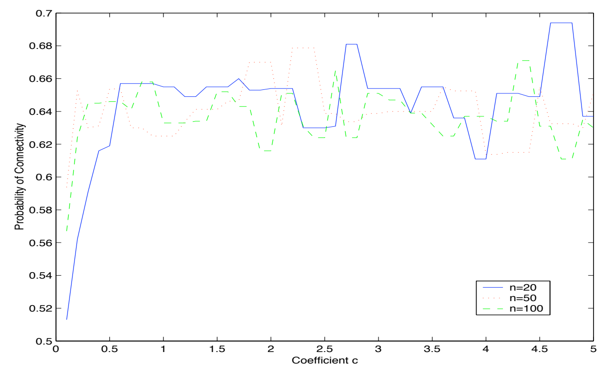


Fig. 11. Phase transition in probability of connectivity when $p(n)R^2(n) = c\frac{\log n}{n}$ and $p(n) = \frac{1}{n}$.

cycling by $R'(n)$. Then,

$$\frac{R'^2(n)}{R^2(n)} = \frac{\log(np(n))}{p(n)\log n}. \qquad (15)$$

Suppose $p(n) = \frac{1}{n^\beta}$, $\beta < 1$ for simplification. Eqn. (15) is

$$\frac{R'^2(n)}{R^2(n)} = \frac{n^\beta \log(n^{1-\beta})}{\log n} = n^\beta(1-\beta). \qquad (16)$$

Figure 12 evaluates Eqn. (15) for some n. As shown in Figure 12, the ratio of $R'(n)$ to $R(n)$ reaches to 1 as β approaches to 0. This is obvious because $p(n)$ decreases at slower rate as n increases, which allows enough nodes to be on without increasing $R'(n)$. At extreme case, $R'(n) = R(n)$ when $\beta = 0$. As β increases, the ratio becomes large. It reaches its maximum when $\beta = 1 - \frac{1}{\log n}$. β approaches to 1 as n goes to ∞.

IX. Conclusion

In this report we studied the asymptotic connectivity of a low duty-cycled wireless sensor network

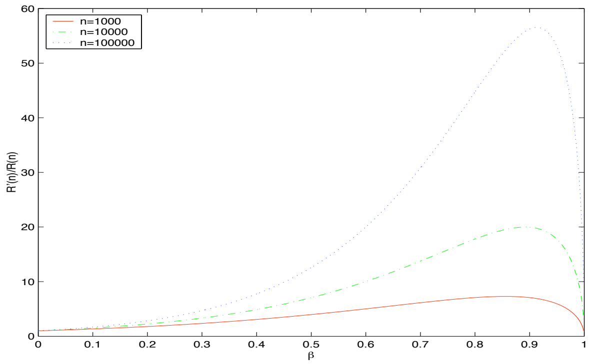


Fig. 12. Ratio of $R'(n)$ to $R(n)$ in terms of β.

where sensor nodes are randomly duty-cycled according to a fixed active probability. We derived the sufficient and necessary conditions for the network to be connected as the number of node grows to infinity. These conditions are in the form of the joint scaling behavior of the number of nodes in the network as well as the active probability. Thus such results reveal how duty-cycling should be scaled as the network gets denser in order to maintain network connectivity.

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