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## TWO THEOREMS IN GRAPH THEORY

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*Communicated by N. E. Steenrod, July 8, 1957*

*Introduction.*—Given an unoriented graph (or 1-dimensional regular complex), let  $X$  be the set of all its vertices and  $U$  be the set of all its edges. When the graph is finite, the following problems arise:

*Problem 1:* A set  $A \subset X$  is said to be *internally stable* if  $x \in A, y \in A$  implies  $(x, y) \notin U$ . The symbol  $|A|$  will denote the number of elements of  $A$ . Construct an internally stable set  $A$  such that  $|A|$  is maximum.

*Problem 2:* A set  $B \subset X$  is said to be a *cover* if every edge of  $U$  is adjacent to at least one vertex in  $B$ . Construct a cover with the minimum number of elements.

*Problem 3:* A set of edges  $V \subset U$  is said to be a *matching* if two edges of  $V$  have no vertex in common. Construct a matching with the maximum number of elements.

A particular case of Problem 1 is the chess problem of Gauss: Put eight queens on the board such that no one can take any other. In  $n$ -person game theory, if the graph of domination is symmetrical, a maximum internally stable set turns out to be a maximum *solution* (in the von Neumann-Morgenstern sense<sup>1</sup>), and the more usual case can be solved by means of the Grundy functions.<sup>2</sup>

Problem 2 is the set theoretic dual of Problem 1, since the complement of an internally stable set is a cover, and conversely. Particular cases of Problem 3 are the problem of distinct representatives (P. Hall<sup>3</sup>) and the problem of Petersen (D. König<sup>4</sup>). In the case where the graph is bipartite, Problem 3 has been solved by algebraic methods by O. Ore,<sup>5</sup> and an efficient algorithm has been given by H. Kuhn.<sup>6</sup> Unfortunately, the linear programming duality used by H. Kuhn no longer subsists when the graph is not bipartite. (Note that Problem 2 is the linear program dual to Problem 3 in the bipartite case.) In view of solving the general case, this paper states two theorems: Theorem 1 gives a necessary and sufficient condition for recognizing whether a matching is maximum and provides an algorithm for Problem 3, while Theorem 2 yields an algorithm for Problems 1 and 2.

*The Theorems.*—Consider a graph  $G = (X, U)$  with a matching  $V_0$ ; if  $u \in V_0$  we shall say that edge  $u$  is *strong*, otherwise that  $u$  is *weak*. An *alternating chain* is a chain which does not use the same edge twice and is such that for any two adjacent edges one is strong and the other is weak. A vertex  $x$  which is not adjacent to a strong edge is said to be *neutral*, the set of all neutral points being  $N$ .

We shall also consider a graph  $\bar{G}$  constructed from  $G$  by adding a vertex  $\bar{a}$  and connecting  $\bar{a}$  to every neutral point with a strong edge. If there exists an alternating chain from  $\bar{a}$  to a vertex  $x$ , we shall picture an arrow on the last edge  $(z, x)$ , directed from  $z$  to  $x$ . A vertex  $x (\in N)$  which is not adjacent to a directed edge is said to be *inaccessible*, the set of all inaccessible points being  $I$ . A vertex  $x (\in N)$  adjacent to a weak edge directed to  $x$  and not to a strong edge directed to  $x$  is said to be *weak*, the set of all weak points being  $W$ . A vertex  $x (\in N)$  adjacent to a strong edge directed to  $x$  and not to a weak edge directed to  $x$  is said to be *strong*,

the set of all strong points being  $S$ . A vertex  $x$  ( $\notin N$ ) adjacent to a strong edge directed to  $x$  and to a weak edge directed to  $x$  is said to be *medium*, and the set of all medium points will be designated by  $M$ .

LEMMA 1. *Let  $Y$  be a connected component of the subgraph  $M$ ; if  $\bar{a}$  is inaccessible, there exists in  $\bar{G}$  one strong edge adjacent to  $Y$  and directed to  $Y$  only; all other edges adjacent to  $Y$  are weak and directed from  $Y$  only. Moreover, all vertices not in  $Y$  and connected to  $Y$  by one edge are weak, and  $|Y| \geq 3$ .*

This is a theorem of T. Gallai;<sup>7</sup> a shorter proof is given by Berge.<sup>8</sup>

LEMMA 2. *If  $\bar{a}$  is inaccessible,  $S \cup N$  is internally stable.*

(Immediate.)

LEMMA 3. *If  $\bar{a}$  is inaccessible,  $M = \phi$  and  $I = \phi$ , then  $S \cup N$  is a maximum internally stable set,  $W$  is a minimum cover, and  $V_0$  is a maximum matching.*

From Lemma 2,  $S \cup N$  is internally stable, hence  $W = X - (S \cup N)$  is a cover. For every cover  $C$  and for every matching  $V$ , one has  $|C| \geq |V|$ ; as  $|W| = |V_0|$ , the cover  $W$  is minimum and the matching  $V_0$  is maximum.

LEMMA 4. *Let  $Z$  be a connected component of the subgraph  $I$ ; if  $\bar{a}$  is inaccessible, all edges adjacent to  $Z$  are weak and undirected; moreover, all vertices not in  $Z$  connected to  $Z$  by an edge are weak, and  $|Z| \geq 2$ .*

(Immediate.)

LEMMA 5. *If  $|N| \leq 1$ ,  $V_0$  is a maximum matching.*

This follows from the fact that  $|X| = 2|V_0| + |N|$ .

LEMMA 6. *If  $A \subset X$ , let  $G_A$  be the graph constructed from  $G$  by shrinking  $A$  into a single vertex  $a_A$ , having as adjacent edges the adjacent edges of  $A$ . If the original strong edges constitute a maximum matching for the subgraph  $A$ , and for  $G_A$ , then  $V_0$  is a maximum matching for  $G$ .*

This is easy to see by an induction on the number of elements of  $A$ .

THEOREM 1. *A matching  $V$  is maximum if and only if there does not exist an alternating chain connecting a neutral point to another neutral point.*

If there existed an alternating chain  $W = (u_1, u_2, \dots, u_k)$  connecting a neutral point  $a$  to a neutral point  $a'$  different from  $a$ ,  $(V - W) \cup (W - V)$  would be a matching with more elements than  $V$ , and  $V$  would not be maximum.

Conversely, let us prove that, if such a chain does not exist,  $V$  is maximum; the proposition being obvious when the graph has one or two edges, we shall assume that the proposition is true for any graph having fewer than  $m$  edges, and we shall prove it for a graph  $G$  of  $m$  edges. One can assume that  $G$  is connected.

From Lemma 5, one can assume  $|N| > 1$ ; from Lemma 3, one can also assume that either  $M \neq \phi$  or  $I \neq \phi$ .

1. If  $M \neq \phi$ , let  $Y$  be a connected component of the subgraph  $M$ ; the graph  $G_Y$  constructed from  $G$  by shrinkage satisfies the conditions of the theorem (Lemma 1); as it has at least one edge less than  $G$ , the strong edges constitute a maximum matching for  $G_Y$ . On the other hand, the subgraph  $Y$  has only one neutral point (Lemma 1) and therefore its strong edges constitute a maximum matching. Thus, from Lemma 6,  $V_0$  is a maximum matching for  $G$ .

2. If  $I \neq \phi$ , let  $Z$  be a connected component of subgraph  $I$ , and consider the graph  $G_Z$ . The vertex  $a_Z$  is a neutral point, connected only with weak points. No alternating chain leads from a point of  $N$  to  $a_Z$ . As  $G_Z$  satisfies the conditions of the theorem,  $G_Z$  admits its strong edges as a maximum matching. On the

other hand, the subgraph  $Z$ , having no neutral points, admits its strong edges as a maximum matching; therefore,  $V_0$  is a maximum matching for  $G$ .

**THEOREM 2.** *Let  $C_Y$  (resp.  $C_Z$ ) be any minimum cover for the subgraph generated by a connected component  $Y$  of  $M$  (resp.  $Z$  of  $I$ ). If there does not exist an alternating chain connecting a neutral point to another neutral point, the set*

$$C = W \cup \bigcup_Y C_Y \cup \bigcup_Z C_Z$$

*is a minimum cover for  $G$ .<sup>9</sup>*

Every vertex which is connected by an edge to a component  $Y$  is a weak point (Lemma 1); every vertex which is connected by an edge to a component  $Z$  is a weak point (Lemma 4). Therefore  $C$  is a cover for  $G$ . As  $C$  is a minimum cover for the graph  $G'$  deduced from  $G$  by removing all edges connecting a weak vertex to a medium or inaccessible vertex (Lemma 3),  $C$  is also a minimum cover for  $G$ .

Theorem 1 suggests the following procedure for solving Problem 3; Construct a maximal matching  $V$ , and determine whether there exists an alternating chain  $W$  connecting two neutral points. (The procedure is known.) If such a chain exists, change  $V$  into  $(V - W) \cup (W - V)$ , and look again for a new alternating chain; if such a chain does not exist,  $V$  is maximum.

Theorem 2 gives an algorithm for Problem 2, hence for Problem 1.

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<sup>1</sup> J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior* (Princeton, N. J.: Princeton University Press, 1944).

<sup>2</sup> C. Berge, "Fonctions de Grundy d'un graphe infini," *Compt. rend. Acad. Sci. Paris*, **242**, 1604, 1956; C. Berge and M. P. Schützenberger, "Jeux de Nim et solutions," *Compt. rend. Acad. Sci. Paris*, **242**, 1672-1674, 1956.

<sup>3</sup> P. Hall, "On Representatives of Subsets," *J. London Math. Soc.*, **10**, 26-30, 1935.

<sup>4</sup> D. König, *Theorie der endlichen und unendlichen Graphen* (Leipzig, 1936).

<sup>5</sup> O. Ore, "Graphs and Matching Theorems," *Duke Math. J.*, **22**, 625-639, 1955.

<sup>6</sup> H. Kuhn, "The Hungarian Method for the Assignment Problem," *Naval Research Logistics Quart.*, **2**, 83-97, 1955.

<sup>7</sup> T. Gallai, "On Foundation of Graphs," *Acta Math. Hung.*, **1**, 133-153, 1950.

<sup>8</sup> C. Berge, *Théorie des Graphes* (Dunod publ., in preparation).

<sup>9</sup> R. Z. Norman and Michael O. Rabin proved independently a similar theorem (cf. An algorithm for a minimum cover of a graph, [Abstract], Washington, D. C. meeting of the A.M.S., October 26, 1957) which could be in some sense a dual of this result and which yields an algorithm for the following problem: construct a set  $C$  of edges such that every vertex is incident to an edge of  $C$ , and which has a minimum number of edges.