Chapter XXIII

APPLICATION OF THE SIMPLEX METHOD TO A TRANSPORTATION PROBLEM [1]

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A number of years before the Air Force generalized the work of Leontief to make it applicable to highly dynamic situations, Hitchcock [1941] and Koopmans [XIV, 1947] independently considered an interesting special case: A homogeneous product is to be shipped in the amounts $a_1, a_2, \cdots, a_m$, respectively, from each of m shipping *origins* and received in amounts $b_1, b_2, \cdots, b_n$, respectively, by each of n shipping *destinations*. The cost of shipping a unit amount from the ith origin to jth destination is c_{ij} and is known for all combinations (i, j). The problem is to determine the amounts x_{ij} to be shipped over all routes (i, j) so as to minimize the total cost of transportation. In Table I it is clear that x_{ij} must be chosen so that the rows sum to the marginal

Table I. Program of Shipments

	$i \backslash j$	Destinations (1)	(2)	$\cdots$	(n)	Total
Origins	(1)	x_{11}	x_{12}	$\cdots$	x_{1n}	a_1
	(2)	x_{21}	x_{22}	$\cdots$	x_{2n}	a_2
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	(m)	x_{m1}	x_{m2}	$\cdots$	x_{mn}	a_m
Total		b_1	b_2	$\cdots$	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

[1] The author is indebted to Emil D. Schell for assistance in preparing earlier versions of this chapter.

totals a_i and the columns to b_j. The basic relations that must be satisfied are

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i = 1, 2, \cdots, m), \tag{1}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j = 1, 2, \cdots, n), \tag{2}$$

$$x_{ij} \geqq 0, \tag{3}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} = \text{min}. \tag{4}$$

The linear programming problem concerns itself with minimization (maximization) of a linear form whose variables satisfy a system of linear inequalities. Usually in practice this problem is encountered in the above standard form, namely, as the minimization of a linear form of *nonnegative* variables subject to a system of linear *equalities*.

1. Application of the Simplex Method

According to the general theory [XXI], if there are k independent equations in l variables, a solution (provided one exists) which minimizes the linear form can be obtained that involves at most k variables with positive value while the remaining $l - k$ variables vanish. Chapter XXI establishes this under the condition that every determinant of kth order is nonvanishing. This condition is not satisfied in the transportation case; however, an earlier version of this chapter contains a direct proof of this theorem which can be slightly altered to remove this restriction [XXI, Theorem A; see also XV]. The method of proof is to show that, if any feasible solution involves more than k variables with positive values, the number can be reduced.

It is not difficult to show that the $m + n$ equations (1) and (2) constitute $m + n - 1$ independent equations in mn unknowns. Thus the minimizing solution requires at most $m + n - 1$ routes with positive shipments.

It is useful to reformulate the transportation problem in terms of a system of activities that have various items in common. The activity of shipping the homogeneous product from i to j will be denoted by A_{ij}. To sustain a *unit level* of this activity, one unit of the product at the ith origin is required as input, and one unit at the jth destination will be made available as output. We shall by convention use + to indicate flow toward an activity and − to indicate flow away from an activity of an item. Thus, if m "origin" items and n "destination" items are defined, a unit of activity A_{ij} is characterized by a vector with $+1$ (input) for origin item i, -1 (output) for destination item j, and 0 for all other items. We shall use the same symbol to denote an activity A_{ij} and the vector associated with unit amounts of the activity A_{ij}. The elements of the vector A_{ij} are shown in Table II.

TABLE II. ELEMENTS OF A_{ij}

Item	$(A_{ij}) = (\xi_i) + (\eta_j)$			B_0
Origin				
1	0	0	0	a_1
2	0	0	0	a_2
...	...	...	...	...
i	+1	+1	0	a_i
...	...	...	...	...
m	0	0	0	a_m
Destination				
1	0	0	0	$-b_1$
2	0	0	0	$-b_2$
...	...	...	...	...
j	−1	0	−1	$-b_j$
...	...	...	...	...
n	0	0	0	$-b_n$

It will be noted that, if a dummy activity, ξ_i, is defined with $+1$ for origin item i and 0 elsewhere, and similarly η_j is defined with -1 for destination item j and 0 elsewhere, all mn activities have a simple representation in terms of this basic set of $m + n$ dummy activities:

$$A_{ij} = \xi_i + \eta_j. \tag{5}$$

A feasible solution consisting of $m + n - 1$ combinations is easily obtained provided only $a_i \geqq 0$, $b_j \geqq 0$, and $\sum a_i = \sum b_j$. For example, A_{11} can be chosen first and x_{11} units of this activity performed, where $x_{11} = \min(a_1, b_1)$. If $a_1 \leqq b_1$, then obviously all other x_{ij} in the first row of Table I vanish, and the corresponding $n - 1$ activities are excluded from the feasible solution. Deleting the row and replacing b_1 by $b_1 - a_1$ reduces the rectangular array in Table I by one row. (If $a_1 > b_1$, the other elements in the column would be deleted.) Continuing this process, a row or column will be deleted and one activity selected at each step until only one row or column is left. Thus, if, for example, in k steps $m - 1$ rows and $k - (m - 1)$ columns have been determined, the remaining $n - [k - (m - 1)]$ activities in the last row will be used to complete the set of activities in the feasible solution. Accordingly, $k + n - [k - (m - 1)] = m + n - 1$ activities have been chosen. The possibility of one or more $x_{ij} = 0$ in the set of $m + n - 1$ activities is not excluded.

Moreover, A_{11} is followed by A_{12} (or A_{21}) and A_{12} followed by A_{13} (or A_{22}), etc. In general, A_{ij} is followed by $A_{i(j+1)}$ or $A_{(i+1)j}$. It is thus a simple matter to express the dummy activities in terms of the activities of the feasible solution. If $A_{11}, A_{12}, A_{22}, A_{32}, A_{33}$, etc., appear in the solution, then we obtain, by taking differences of activities as they are generated,

$$
\begin{aligned}
\eta_2 - \eta_1 &= A_{12} - A_{11}, \\
\xi_2 - \xi_1 &= A_{22} - A_{12}, \\
\xi_3 - \xi_2 &= A_{32} - A_{22}, \\
\eta_3 - \eta_2 &= A_{33} - A_{32},
\end{aligned}
\tag{6}
$$

so that $\xi_2 - \xi_1, \xi_3 - \xi_2, \cdots, \xi_m - \xi_{m-1}$ will be determined in turn, as well as $\eta_2 - \eta_1, \cdots, \eta_n - \eta_{n-1}$. We may thus directly express any ξ_i and η_j in terms of the activities of the feasible solution. Denoting the activities of the feasible solution by $B_1, B_2, \cdots, B_{m+n-1}$, and making use of the relation $\eta_1 = B_1 - \xi_1$, it is a straightforward matter of summing differences on either ξ_i or η_j to obtain a solution of ξ_i or η_j as a linear combination of the vectors $B_1, B_2, \cdots, B_{m+n-1}$ and ξ_1:

$$
\begin{aligned}
\xi_i &= \xi_1 + \sum_{k=1}^{m+n-1} \lambda_{ik} B_k \qquad (i = 1, 2, \cdots, m), \\
\eta_j &= -\xi_1 + \sum_{k=1}^{m+n-1} \mu_{jk} B_k \qquad (j = 1, 2, \cdots, n),
\end{aligned}
\tag{7}
$$

where λ_{ik} and μ_{jk} are constants. Moreover, from (5) any A_{ij} is given by

$$
A_{ij} = \sum_{k=1}^{m+n-1} (\lambda_{ik} + \mu_{jk}) B_k. \tag{8}
$$

The fundamental approach of the simplex technique is to express all activities in the system in terms of a basic set of activities constituting a feasible solution. This has just been done. There are thus two ways to accomplish an activity A_{ij}, either directly or indirectly as a linear combination of activities B_k. In (8), however, the coefficients can be positive or negative. This is interpreted to mean that one unit of A_{ij} can be done by doing $(\lambda_{i1} + \mu_{j1})$ units of B_1, $(\lambda_{i2} + \mu_{j2})$ units of B_2, etc. When the coefficient of B_1 is negative, it means to decrease the number of units of B_1 by this amount, *if possible*, in some system in which $B_1, B_2, \cdots, B_{m+n-1}$ are being performed at some positive number of units.

The next step of the method is to "cost" the direct versus the indirect way of doing one unit of A_{ij}. The direct cost of one unit of A_{ij} is c_{ij}, the direct costs of $B_1, B_2, \cdots, B_{m+n-1}$ will be denoted by $c_1, c_2, \cdots, c_{m+n-1}$; i.e., if $B_1 = A_{11}$, then $c_1 = c_{11}$. The indirect cost of A_{ij} will be denoted by $\bar{c}_{ij}$,

$$\bar{c}_{ij} = \sum_{k=1}^{m+n-1} (\lambda_{ik} + \mu_{jk})c_k = u_i + v_j, \tag{9}$$

where u_i and v_j "cost" the dummy activities ξ_i and η_j,

$$\begin{aligned} u_i &= \lambda_{i1}c_1 + \cdots, + \lambda_{i,\,m+n-1}c_{m+n-1} \quad (i = 1, 2, \cdots, m), \\ v_j &= \mu_{j1}c_1 + \cdots, + \mu_{j,\,m+n-1}c_{m+n-1} \quad (j = 1, 2, \cdots, n). \end{aligned} \tag{10}$$

The general theory states that if

$$c_{ij} < \bar{c}_{ij} \tag{11}$$

it pays to introduce A_{ij} and to drop one of the activities $B_1, B_2, \cdots, B_{m+n-1}$ from the feasible solution. Which one to drop will now be discussed.

Let $x_1, x_2, \cdots, x_{m+n-1}$ be the number of units of $B_1, B_2, \cdots, B_{m+n-1}$ in the feasible solution; then

$$x_1B_1 + x_2B_2 + \cdots + x_{m+n-1}B_{m+n-1} = B_0 \qquad (x_i > 0), \tag{12}$$

where B_0 is the column vector $a_1, \cdots, a_m, -b_1, \cdots, -b_n$ (see Table II). It will be noted that x_i is *assumed positive*. The case where one or more $x_i = 0$ will be considered degenerate and will be discussed later. The total cost of the solution given by (12) will be denoted by z_0;

$$x_1c_1 + x_2c_2 + \cdots, + x_{m+n-1}c_{m+n-1} = z_0. \tag{13}$$

Assume $c_{ij} < \bar{c}_{ij}$ for some A_{ij}, and rewrite (8) and (9) as

$$A_{ij} - (\nu_1B_1 + \nu_2B_2 + \cdots + \nu_{m+n-1}B_{m+n-1}) = 0, \tag{14}$$

$$c_{ij} - (\nu_1c_1 + \nu_2c_2 + \cdots + \nu_{m+n-1}c_{m+n-1}) = -(\bar{c}_{ij} - c_{ij}), \tag{15}$$

where

$$\nu_k = \lambda_{ik} + \mu_{jk}. \tag{16}$$

By multiplying equation (14) by θ and adding this to (12), other feasible solutions are obtained provided $\theta > 0$ and θ is not so large that any coefficient of B_i is negative. By multiplying (15) by θ and adding this to (13), the corresponding cost, z_1, of the new feasible solution is obtained;

$$z_1 = z_0 - \theta(\bar{c}_{ij} - c_{ij}), \tag{17}$$

which by (11) is clearly *less* than z_0.

Now there is a very simple rule for evaluating the largest value of θ. Referring back to (6), (7), and (8) it will now be shown that the values of λ_{ik} and μ_{jk} are either 0, +1, or −1. If $\lambda_{ik} = +1$ for any B_k, then $\mu_{jk} = 0$ or -1, because in (6) the A_{ij} does not appear with the same sign for differences involving ξ and η. To put it another way, coefficients in (8) are either 0, +1, or −1. Moreover, if (8) is used to eliminate any B_k from (7) in order to express ξ_i and η_j in terms of the remaining B_k and the new A_{ij}, it is clear from the structure of A_{ij} that the same properties will hold after the elimination. Thus all coefficients in (14) are +1, −1, or 0. Any $\nu_i = +1$ in (14) will automatically place a restriction on the size of θ. The maximum θ is thus the minimum x_i in (12) whose corresponding $\nu_i = +1$ in (14). Therefore

$$\theta = \min x_i, \qquad \nu_i = +1. \tag{18}$$

If the minimum occurs for $i = k$, then B_k will be eliminated, and the new solution consists again of $m + n - 1$ activities.

The new feasible solution has x_i increased by θ for $\nu_i = -1$, decreased by θ for $\nu_i = +1$, untouched for $\nu_i = 0$. There will be θ units of A_{ij} introduced. The cost of the new solution is given by (17). The "cost" of the dummy activities ξ_i and η_j given by (10) will be decreased or increased by $+(\bar{c}_{ij} - c_{ij})$, or remain unchanged accordingly as the coefficients of B_k in (7) or of c_k in (10) appear equal to +1, −1, or 0, respectively. It should be noted that, if any u_i is increased, no v_j can be increased and conversely.

The selection of A_{ij} to improve the feasible solution depended on $c_{ij} < \bar{c}_{ij}$. The improvement in z, however, may be small or large depending on which A_{ij} is chosen. It has been found empirically that selection of A_{ij} such that

$$c_{ij} - \bar{c}_{ij} = \max \quad (i = 1, \cdots, m; j = 1, \cdots, n) \tag{19}$$

will seldom introduce or eliminate an activity that is not in the final solution. Other criteria for selection of A_{ij} are discussed in Chapter XXI.

The process described is iterated, each iteration producing a new feasible solution involving $m + n - 1$ activities. For each iteration the value of z, the total cost, is decreased. In a finite number of steps an optimum solution is obtained, since the solution at each iteration is unique, and there is only a finite number of ways to choose a basic set of $m + n - 1$ activities.

2. The Case of Degeneracy

If for any iteration several B_k are eliminated, the general rule is to treat only one as eliminated and leave the others formally in the solution even though they appear with zero weight. A criterion will now be developed for the determination of the B_k with zero weight to be dropped from the basic solution. This is necessary, because it is not known whether an *arbitrary* selection will lead to a decrease in total cost in a finite number of iterations. In empirical examples *arbitrary* selection has proved to be a good working rule. However, for a slight amount of additional effort, one can protect oneself against possible failure of the method in degenerate cases. The fundamental idea is that by slight modifications of the marginal totals, a_i and b_j, in Table I, degeneracy can be avoided in a family of equations whose marginal totals differ uniformly from the corresponding a_i and b_j by less than any desired ϵ.[2]

Any basic solution consists of $m + n - 1$ combinations, and, in case of degeneracy, it involves one or more combinations with zero weights. When this occurs, it implies that a partial sum of the a_i's equals a partial sum of the b_j's. The proof can be argued as follows:

Let $m \geqq n$; then there is at least one row in Table I which contains exactly one B_k from the basic solution (there must always, of course, be one or more for each column or row). Otherwise the number of A_{ij} in the basic solution would be at least $2m$, which would require $2m \leqq m + n - 1$, or $m \leqq n - 1$, i.e., a contradiction. Thus one of the rows yields $x_{ij} = a_i$. Deleting the ith row and replacing b_j by $b_j - a_i$, the process may be repeated with the reduced array. With each iteration the new row or column totals differ from the previous ones by one difference, $a_i - b_j$ or $b_j - a_i$; in terms of the original a_i and b_j's the new a_i and b_j are differences of partial sums of the original a_i and b_j. Thus, if at any stage an $x_{ij} = 0$, this implies the vanishing of both the new a_i and b_j.

[2] The *specific* way to alter the marginal totals by ϵ to avoid "degeneracy" problems has been introduced in this version, although it was indicated as possible in earlier papers. This extension was stimulated by Robert Dorfman and Merrill Flood, as well as by Tjalling C. Koopmans and M. L. Slater, the referees of this manuscript, who have insisted that the status of degeneracy be clarified.

Thus degeneracy can be avoided if we can prevent any partial sum of the a_i's equaling a partial sum of the b_j's. Consider a class of problems with unspecified ϵ in which

$$\bar{a}_i = a_i + \epsilon \qquad (i = 1, 2, \cdots, m),$$

$$\text{(20)} \qquad \bar{b}_j = \begin{cases} b_j & (j = 1, 2, \cdots, n-1), \\ b_j + m\epsilon & (j = n). \end{cases}$$

Assume $\epsilon > 0$. It will be shown that there exists an ϵ_0 such that for any ϵ in the range $0 < \epsilon < \epsilon_0$ there can be no partial sum of $\bar{a}_i$'s equal to a partial sum of $\bar{b}_j$'s. There are a finite number of possible equalities of partial sums of $\bar{a}_i$'s to partial sums of $\bar{b}_j$'s. Consider the kth of these possibilities; it will be shown that there exists a range $0 < \epsilon < \epsilon_k$ in which $\sum_k \bar{a}_i \neq \sum_k \bar{b}_j$, where ϵ_k depends on the partial sums in question. The sum of the coefficients of ϵ associated with the $\bar{a}_i$ cannot be equal to the sum of the coefficients of ϵ associated with the $\bar{b}_j$, for the sum on the $\bar{a}$ side has a minimum of 1 and a maximum of $m - 1$, while the sum on the $\bar{b}$ side is either 0 or m. Since the coefficients of ϵ are not equal, by setting $\sum_k \bar{a}_i = \sum_k \bar{b}_j$ we can solve for ϵ. If $\epsilon \leqq 0$, set $\epsilon_k = +\infty$, and if $\epsilon > 0$, set $\epsilon_k = \epsilon$. There are a finite number of ϵ_k's. Let ϵ_0 be the smallest of them; then for $0 < \epsilon < \epsilon_0$, there is no partial sum of $\bar{a}_i$'s equal to a partial sum of $\bar{b}_j$'s.

For any basis, the general solution $\{\bar{x}_k\}$ to $(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_m; \bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n)$ can be represented as the sum of two special solutions; the first, $\{x_k\}$, for $(a_1, a_2, \cdots, a_m; b_1, b_2, \cdots, b_n)$, the second, $\{x'_k\}$, for the coefficients of ϵ, where $k = 1, 2, \cdots, m + n - 1$. Thus

$$\text{(21)} \qquad \bar{x}_k = x_k + x'_k\epsilon \qquad (k = 1, 2, \cdots, m + n - 1).$$

In the shift from one basis to the next, the use of $\bar{x}_k$ constitutes a small additional effort over just working with x_k alone. This device may now be used to resolve all ties (i.e., equalities, that give rise to degeneracy). Suppose, for example, that in (18) $\theta = x_k = x_l$; then the min $[\bar{x}_k, \bar{x}_l]$ is chosen by comparing the value of x'_k with x'_l.[3]

[3] It has been proved by A. Orden that the following procedure for assigning a fixed value to ϵ removes degeneracy and is convenient for computation. Let δ equal the least significant digit in the shipments a_i and b_j. Take ϵ equal to the largest significant digit in $\delta/2m$. This ϵ used in (20) permits no equalities of partial sums. All computations are done on the basis of $\bar{a}_i$ and $\bar{b}_j$. These artificial shipments have more significant digits than the original problem. Upon completion of the computations, the final $\bar{x}_{ij}$ values are rounded off to the same number of significant digits as in the original a_i and b_j, and the results are then an exact minimum cost solution to the problem.

A solution which minimizes the total cost will be reached in a finite number of iterative steps because the removal of degeneracy by the ϵ technique makes it impossible for a basis to appear more than once. With degeneracy removed, the total cost must decrease at each stage of iteration, which would not be true if a basis were to recur. Since the number of possible bases is finite, a minimum cost solution must be reached in a finite number of iterations.

3. Rules for Computing an Optimum Solution

(a) Construct a unit cost table giving the cost, c_{ij}, to ship a unit amount from shipping origin i to shipping destination j. In the example (Table III), the cost to ship from origin (2) to destination (1) is 5; from origin (2) to destination (4) is -1. No interpretation is given to negative cost, except to show that there is no restriction on the sign of c_{ij}. A constant amount may be added or subtracted uniformly from all c_{ij} without affecting the values of x_{ij} appearing in the solution. In Table IV the total amount to be shipped is 13. If all c_{ij} were increased by 2, the total cost of transportation would be increased by $2 \cdot 13 = 26$.

Table III. Direct Unit Costs, c_{ij}

		Destinations				
	$i \backslash j$	(1)	(2)	(3)	(4)	(5)
Origins	(1)	3	2	1	2	3
	(2)	5	4	3	-1	1
	(3)	0	2	3	4	5

Table IV. Amounts to be Shipped

		Destinations					Total
	$i \backslash j$	(1)	(2)	(3)	(4)	(5)	$\bar{a}_i$
Origins	(1)	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	$1 + \epsilon$
	(2)	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	$5 + \epsilon$
	(3)	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	$7 + \epsilon$
Total	$\bar{b}_j$	3	3	3	2	$2 + 3\epsilon$	$13 + 3\epsilon$

(b) The basic problem is illustrated in the shipping table of the example (Table IV). Unknown values, $x_{ij} \geqq 0$, are to be selected so that the row totals sum to $a_1 = 1$, $a_2 = 5$, $a_3 = 7$, and the column totals to $b_1 = b_2 = b_3 = 3$, $b_4 = b_5 = 2$ in such a manner that $\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij}$ is a minimum. For computation purposes Table IV should be set up leaving the boxes for x_{ij} blank. To avoid degeneracy, a_i and b_j are considered the limits of $\bar{a}_i$ and $\bar{b}_j$ as $\epsilon \to +0$, where (1) $\bar{a}_i = a_i + \epsilon$ for $i = 1, 2, \cdots, m$, and (2) $\bar{b}_j = b_j$ for $j = 1, 2, \cdots, n - 1$ and $\bar{b}_j = b_j + m\epsilon$ for $j = n$.

(c) An arbitrary basic solution is obtained (Table V) by assigning a

TABLE V. ARBITRARY BASIC SOLUTION, $\bar{x}_{ij}$

		Destinations					Total
	$i \backslash j$	(1)	(2)	(3)	(4)	(5)	$\bar{a}_i$
Origins	(1)	$1 + \epsilon$					$1 + \epsilon$
	(2)	$2 - \epsilon$	$3 + 0\epsilon$	$0 + 2\epsilon$			$5 + \epsilon$
	(3)			$3 - 2\epsilon$	$2 + 0\epsilon$	$2 + 3\epsilon$	$7 + \epsilon$
Total	$\bar{b}_j$	3	3	3	2	$2 + 3\epsilon$	$13 + 3\epsilon$

value, $\bar{x}_{11} = \min(\bar{a}_1, \bar{b}_1)$ as $\epsilon \to +0$. In the example $3 + 0\epsilon > 1 + \epsilon$. If $\bar{a}_1$ is minimum, all other $\bar{x}_{ij}$ in the first row are zero; if $\bar{b}_1$ is minimum, all other $\bar{x}_{ij}$ in the first column are zero. Deleting then, the evaluated row or column, the procedure is now repeated with the remaining rows or columns where the marginal totals are reduced by the evaluated part.

Actually there is no need to carry along the ϵ part of this solution unless at some stage there is an equality when taking a minimum. In the example such an equality took place in the third stage in evaluating $x_{22} = 3$. Thus there appears to be a choice whether to have (2, 3) or (3, 2) included as next point in the basis. However, by going back and including the ϵ part of the solution, it is clear that (2, 3) is the next combination to be introduced into the basis.

(d) Step 1, part I of the iterative process, consists in determining the indirect unit cost table, $\bar{c}_{ij}$, associated with the basis. For any (i, j) appearing in the basis, $\bar{c}_{ij} = c_{ij}$. Any other $\bar{c}_{ij}$ is obtained through the relation $\bar{c}_{ij} = u_i + v_j$. In the example, start with any c_{ij} from a basis (e.g., $c_{ij} = c_{34}$). Arbitrarily set $u_i = c_{ij}$ and $v_j = 0$; thus

$$u_3 = c_{34}, \qquad v_4 = 0.$$

Consider next all c_{kl} from the basis that have a subscript in common with c_{34}; these are c_{33} and c_{35}. From this v_3 and v_5 can be evaluated by

$$v_3 - v_4 = c_{33} - c_{34},$$

$$v_5 - v_4 = c_{35} - c_{34}.$$

Consider next all c_{kl} that have subscripts in common with c_{33} and c_{35}; this set consists only of c_{23}, whence

$$u_2 - u_3 = c_{23} - c_{33}.$$

Consider next the c_{kl} with subscripts in common with c_{23}; these are c_{21} and c_{22}, whence

$$v_1 - v_3 = c_{21} - c_{23},$$

$$v_2 - v_3 = c_{22} - c_{23}.$$

Finally, the only c_{kl} with subscripts in common with c_{21} and c_{22} is c_{11}, whence

$$u_1 - u_2 = c_{11} - c_{21}.$$

The indirect unit cost table, $\bar{c}_{ij} = u_i + v_j$, may now be formed. In Table VI, step 1, the (i, j) combinations occurring in the basis are given in bold-face type; for these, $\bar{c}_{ij} = c_{ij}$.

(e) Compare the indirect cost table, $\bar{c}_{ij} = u_i + v_j$, with the direct costs, c_{ij}, in Table III, and form

$$M = \max (\bar{c}_{ij} - c_{ij}) = \bar{c}_{kl} - c_{kl}.$$

There are two possibilities: $M > 0$ or $M = 0$. If $M > 0$, select any combination (k, l) such that $\bar{c}_{kl} - c_{kl} = M$. This means that as many units as possible, $\theta = \theta_1$, of combination (k, l) are to be introduced into the transportation schedule, and the remainder is to be made up from combinations in the basis. If $M = 0$, it means that the basis represents the final solution, and no units of any other combination are to be introduced. In Table VI, step 1, the element $\bar{c}_{24}$ is boxed to indicate $M = \bar{c}_{24} - c_{24} = 5$. This is an arbitrary selection since, also, $M = \bar{c}_{31} - c_{31} = 5$.

(f) Step 1, part II of the iterative process, consists in determining the solution of the transportation problem in terms of the combinations occurring in the basis under one of two assumptions: If $M = \bar{c}_{kl} - c_{kl} > 0$, an unknown number of units $\theta = \theta_1$, of combination (k, l), will be assumed to occur (if $M = 0$, no units of any other combination will be assumed to occur). The shipping table is solved in terms of the basis by seeking a row or column in which only one element appears in the

TABLE VI. ITERATIVE PROCESS OF COST MINIMIZATION

Step 1

$\bar{c}_{ij}$

3	2	1	2	3
5	4	3	4	6
5	4	3	4	5

x_{ij}

1					1
2	3	$0-\theta_1$	θ_1		5
		$3+\theta_1$	$2-\theta_1$	2	7
3	3	3	2	2	13

Step 2

$\bar{c}_{ij}$

3	2	−4	−3	−2
5	4	−2	−1	0
10	9	3	4	5

$\bar{x}_{ij}$

$1 \quad +\epsilon$					$1+\epsilon$
$2-\theta_2-\epsilon$	3		$0+\theta_2+2\epsilon$		$5+\epsilon$
θ_2		3	$2-\theta_2-2\epsilon$	$2+3\epsilon$	$7+\epsilon$
3	3	3	2	$2+3\epsilon$	$13+3\epsilon$

Step 3

$\bar{c}_{ij}$

3	2	6	−3	8
5	4	8	−1	10
0	−1	3	−6	5

x_{ij}

1					1
$0-\theta_3$	3		2	$+\theta_3$	5
$2+\theta_3$		3		$2-\theta_3$	7
3	3	3	2	2	

Step 4

$\bar{c}_{ij}$

3	11	6	6	8
−4	4	−1	−1	1
0	8	3	3	5

x_{ij}

$1-\theta_4$	$+\theta_4$				1
	$3-\theta_4$		2	$0+\theta_4$	5
$2+\theta_4$		3		$2-\theta_4$	7
3	3	3	2	2	

TABLE VI (*Continued*)

Step 5

$\bar{c}_{ij}$

6	2	−3	−3	−1
−4	4	−1	−1	1
0	[8]	3	3	5

x_{ij}

	1				1
	$2-\theta_5$		2	$1+\theta_5$	5
3	$+\theta_5$	3		$[1-\theta_5]$	7
3	3	3	2	2	

Step 6

$\bar{c}_{ij}$

0	2	3	−3	−1
2	4	[5]	−1	+1
0	2	2	−3	−1

x_{ij}

	1				1
	$[1-\theta_6]$	$+\theta_6$	2	2	5
3	$1+\theta_6$	$3-\theta_6$			7
3	3	3	2	2	

Step 7

$\bar{c}_{ij}$

0	2	[3]	−1	+1
0	2	3	−1	+1
0	2	3	−1	+1

x_{ij}

	$[1-\theta_7]$	$+\theta_7$			1
		1	2	2	5
3	$2+\theta_7$	$2-\theta_7$			7
3	3	3	2	2	

Step 8

$\bar{c}_{ij}$

−2	0	1	−3	−1
0	2	3	−1	+1
0	2	3	−1	+1

x_{ij}

		1			1
		1	2	2	5
3	3	1			7
3	3	3	2	2	

basis (there is always one such in the larger dimension). Thus $x_{11} = 1$, $x_{22} = 3$, $x_{34} = 2 - \theta_1$, $x_{44} = 2$. These variables are eliminated, and the process is repeated with the remainder. The maximum value for θ_1 that can be introduced equals the minimum x_{ij} in which a term $x_{ij} - \theta_1$ occurs. In the example, min $x_{ij} = x_{23} = 0 = \theta_1$. Thus shipping combination (2, 3) is to be dropped; this combination is the one boxed in the right-hand table. The case where there is a multiple choice of combinations to be dropped is discussed in (g) below.

(g) If $M > 0$ in any step k, the process (d), (e), (f) is repeated for step $k + 1$. A new basis, consisting of all the combinations occurring in the basis for step k, is formed by deleting the boxed combination in the x_{ij}-table and introducing the boxed combination occurring in the c_{ij}-table. In step 2 there was ambiguity as to the combination (i, j) in our example to be dropped. Thus min $x_{ij} = x_{21} = x_{34} = 2 = \theta_2$. *In this case the ϵ component of $\bar{a}_i$ and $\bar{b}_j$ must be adjoined and a solution in terms of ϵ obtained.* It is not possible with this component to have any ambiguity. Thus $\bar{x}_{34} < \bar{x}_{21}$ for $\epsilon > 0$, and combination (3, 4) is the one to be dropped in step 3. If $M = 0$ in step k, the x_{ij}-table represents the final shipping table.

(h) The total cost of any solution is given by $\sum c_{ij}x_{ij}$. A simple formula for evaluating z from one step to the next is given in Table VII.

TABLE VII. EVALUATING TOTAL COST, z

Step	$z_{t+1} = z_t - M_t\theta_t$	$M = \max \bar{c}_{ij} - c_{ij}$	θ
1	$z_1 = 52$ *	$M_1 = 5$	$\theta_1 = 0$
2	$z_2 = 52$	$M_2 = 10$	$\theta_2 = 2$
3	$z_3 = 32$	$M_3 = 9$	$\theta_3 = 0$
4	$z_4 = 32$	$M_4 = 9$	$\theta_4 = 1$
5	$z_5 = 23$	$M_5 = 6$	$\theta_5 = 1$
6	$z_6 = 17$	$M_6 = 2$	$\theta_6 = 1$
7	$z_7 = 15$	$M_7 = 2$	$\theta_7 = 1$
8	$z_8 = 13$	$M_8 = 0$	

* Evaluated directly, $z = \Sigma c_{ij}x_{ij}$.

4. Concluding Remarks

The basis of the first step is characterized by $\bar{c}_{ij} \geqq c_{ij}$ for all (i, j), so that it is the maximum cost basis. Only two of the seven combinations in this basis were also in the minimum cost basis. Thus, as a minimum, five additional steps or six total would be required to eliminate the five wrong combinations in the basis. It took, however, a total of eight instead of six steps. Unfortunately, on step 1 combination (2, 3) appearing in the final basis was dropped and later had to be reintroduced. Also, combination (1, 2) was introduced in step 4 and later had to be eliminated. Experience on large scale problems indicates that the criterion of using max $(\bar{c}_{ij} - c_{ij})$ to introduce a new combination into a basis seldom selects combinations other than ones required in the final basis, or seldom leads to elimination of a correct combination. This does not mean that theoretical problems could not be "cooked up" where this criterion is weak, but that in practical problems the number of steps has not been far from $m + n - 1$.

The indirect unit cost table, $\bar{c}_{ij} = u_i + v_j$, is equivalent to Koopmans' economic potential function [XIV, Section 2.3 and sequel]. (Table VII of Chapter XIV gives the corresponding notations in these two chapters.) Koopmans uses a linear graph to evaluate the potential function and to determine what combinations to introduce and drop. This device was used also in earlier versions of this chapter but was found difficult to use except in hand computations. The closest thing to a tree or linear graph in this chapter is the method of computing $\bar{c}_{ij}$.