Chapter XIV

A MODEL OF TRANSPORTATION [1]

By Tjalling C. Koopmans and Stanley Reiter

In this chapter we shall apply the model developed in Chapter III to the problem of efficient utilization of movable transportation equipment. After discussing the characteristics of an efficient solution to this problem, we shall indicate how marginal rates of substitution between flows of transported goods on various routes can be derived.

For the sake of definiteness we shall speak in terms of the transportation of cargoes on ocean-going ships. In considering only shipping we do not lose generality of application since ships may be "translated" into trucks, aircraft, or, in first approximation,[2] trains, and ports into the various sorts of terminals. Such translation is possible because all the above examples involve particular types of movable transportation equipment.

The models treated here will be "simplified" in several respects. First, they are static models. We describe the joint output of shipping operations as a set of cargo flows, to be referred to as the transportation *program*, which is assumed to be unchanged in quantity over time. (By measuring cargo flows on each route in shiploads, we need not preclude changes in commodity composition of cargo.) Second, we assume that all ships are of the same type and therefore completely interchangeable in each of their uses.

1. A Model with Two Ports

1.1. *Commodities and activities.* The model which we consider first is further simplified in that we assume only two ports, P and Q. The technology matrix for this model is given in Table I. At each port two

[1] The theory presented in this chapter was originally developed by the former author partly during, but mostly after, his association as statistician with the (British-American) Combined Shipping Adjustment Board and with the British Merchant Shipping Mission in Washington during World War II. The responsibility for this chapter rests, of course, with the authors. For a nonmathematical exposition of this model see T. C. Koopmans [1947], where another illustrative example is also given.

[2] The case of railroad equipment is complicated by the "decomposability" of trains, particularly in regard to locomotives.

TABLE I. TECHNOLOGY MATRIX FOR A TWO-PORT MODEL

Commodities	Unit	Symbol	Activities † Port P Loading $\bar{x}_P$	Port P Dis-charging x_P	Port Q Loading $\bar{x}_Q$	Port Q Dis-charging x_Q	Port P to Q Sailing loaded $\bar{x}_{PQ}$	Port P to Q Sailing empty x_{PQ}	Port Q to P Sailing loaded $\bar{x}_{QP}$	Port Q to P Sailing empty x_{QP}
Final: *										
Cargo transportation										
From P to Q	Shiploads per month	y_{PQ}				1				
From Q to P		y_{QP}		1						
Intermediate: *										
Net appearances of loaded ships										
At P for Q	Ships per month	0	1				-1			
At P from Q		0		-1					1	
At Q for P		0			1				-1	
At Q from P		0				-1	1			
Net appearances of empty ships										
At P	Ships per month	0	-1	1				-1		1
At Q		0			-1	1		1		-1
Primary: *										
Availability of shipping	Ships	$\bar{z}$	$-l_P$	$-d_P$	$-l_Q$	$-d_Q$	$-\bar{s}_{PQ}$	$-s_{PQ}$	$-\bar{s}_{QP}$	$-s_{QP}$
Capacity of port P	Berths	z_P	$-k_P$	$-m_P$						
Capacity of port Q	Berths	z_Q			$-k_Q$	$-m_Q$				

* For the concepts final, intermediate, and primary commodities, see Chapter III, Section 1.

† In units of ships per month.

activities,[3] "loading cargo" and "discharging cargo," are defined. For each *route*, i.e., for each ordered pair of ports (in the present case there are two routes) two activities are defined, "sailing with cargo from P to Q," and "sailing in ballast (i.e., without cargo) from P to Q." Each activity is given by a column of coefficients as in Table I. If a *commodity* (associated with a row in Table I) is not involved in a given activity, the coefficient in that row and column is zero; if the commodity is an input, its coefficient in the given activity is negative; if the commodity is an output of the given activity, its coefficient is positive. The net output of a commodity by an activity is assumed to be equal to the coefficient of that commodity multiplied by the *amount* ($\bar{x}_P$, x_P, etc.) of that activity.

The list of commodities, and the units in which commodities and activities are measured, can be read from Table I. We may point out explicitly, however, that the coefficients indicated by the letters l, d, $\bar{s}$, s, appearing in the "shipping" row of Table I have the dimension "time," measured in months. Thus l_P denotes the fraction of a month required to load a ship at port P. Since all activities are measured in units of "ships per month" we have, in the case of the first activity,

$$(\bar{x}_P \text{ ships/month})(l_P \text{ months}) = \bar{x}_P l_P \text{ ships} \tag{1.1}$$

tied up at any instant of time (more precisely: on the average for a long period of time) in loading at port P. This is the correct dimension of z, the total fleet in use. Similarly, the port capacity coefficients k, m, have the dimension "berth-months per ship."

1.2. *Partial reduction of the technology matrix.* We shall explore what vectors of cargo flows are possible in this model, while further simplifying the model as we go on. Note that we must have

$$\bar{x}_P,\ x_P,\ \bar{x}_Q,\ x_Q,\ \bar{x}_{PQ},\ x_{PQ},\ \bar{x}_{QP},\ x_{QP} \geqq 0, \tag{1.2}$$

since no activity can be carried out "in reverse." (Sailing empty from P to Q is not the negative of sailing empty from Q to P because both activities require the employment of ships.)

Since our model is static, we do not permit accumulation (or decumulation) of stocks of idle ships, loaded or empty, in ports. This is expressed by requiring all net output flows of intermediate commodities to be zero. We shall call a set (y_{PQ}, y_{QP}, $\bar{z}$, $\dot{z}_P$, z_Q) of net commodity flows a *possible point* in the commodity space if the flows in question can be accomplished

[3] For the concepts "activity" and "commodity" see Chapter III, Section 1.4. Our use of these concepts here implies that we ignore the indivisibility of individual ships.

by nonnegative activity levels satisfying this requirement. We note that, for all possible points,

$$(1.3) \qquad y_{PQ}, y_{QP} \geqq 0, \qquad \bar{z}, z_P, z_Q \leqq 0,$$

since the coefficients of final commodities in Table I are all nonnegative and those of primary commodities are all nonpositive.

For some purposes, such as the analysis of what can be done in a limited period with a given fleet and given port facilities, it is useful to regard the flows $\bar{z}$, z_P, z_Q of primary commodities as subject to given capacity limitations,

$$(1.4a) \qquad \bar{z} \geqq \zeta, \qquad \zeta < 0,$$

$$(1.4b) \qquad z_P \geqq \zeta_P, \quad z_Q \geqq \zeta_Q, \qquad \zeta_P, \zeta_Q < 0.$$

Any possible point in the commodity space falling within these limitations will be called an *attainable point.*

Before writing out the net output equations for all commodities, we shall utilize the equations for the loaded ship appearances to simplify the technology matrix. These equations are equivalent to

$$(1.5) \qquad \bar{x}_P = \bar{x}_{PQ} = x_Q; \qquad \bar{x}_Q = \bar{x}_{QP} = x_P.$$

Each pair of equations (1.5) requires that three activities be carried out in equal amounts. Since these activities are so tied to one another, nothing will be changed except the appearance of the technology matrix if we define one new activity, "transporting cargo from P to Q" to include unit amounts of "loading at P," "sailing with cargo from P to Q," and "discharging cargo at Q," and if similarly we define another activity, "transporting cargo from Q to P." The coefficients of the new activities, as given in Table II, are the sums of the corresponding coefficients of the component activities,

$$(1.6) \qquad t_{PQ} = l_P + \bar{s}_{PQ} + d_Q, \qquad t_{QP} = l_Q + \bar{s}_{QP} + d_P.$$

We have thus performed a partial *reduction* [III, Section 3.10] of the technology matrix. Each condensation of activities in this reduction replaces three inequalities of the form $x \geqq 0$ by one such inequality, the three x's being the amounts of the component activities and the new restrictions, $\bar{x}_{PQ} \geqq 0$, $\bar{x}_{QP} \geqq 0$, referring to the amounts of the new activities. Table II gives the partially reduced technology matrix.

As it now stands, the model contains three primary commodities. If capacity limits, ζ, ζ_P, ζ_Q, on their inflows are introduced by (1.4a), (1.4b), any of these limits can, and at least one must, constitute an

TABLE II. PARTIALLY REDUCED TECHNOLOGY MATRIX A FOR A TWO-PORT MODEL

Commodities			Activities *			
			Route P to Q		Route Q to P	
			Transporting cargo	Sailing empty	Transporting cargo	Sailing empty
	Unit	Symbol	$\bar{x}_{PQ}$	x_{PQ}	$\bar{x}_{QP}$	x_{QP}
Final:						
Cargo from P to Q	Shiploads per month	y_{PQ}	1			
Cargo from Q to P		y_{QP}			1	
Intermediate:						
Net appearances of empty ships	Ships per month					
At P		0	−1	−1	1	1
At Q		0	1	1	−1	−1
Primary:						
Availability of shipping	Ships	$\bar{z}$	$-t_{PQ}$	$-s_{PQ}$	$-t_{QP}$	$-s_{QP}$

* In units of ships per month.

effective restriction on the set of attainable points.[4] We shall then simplify our model by assuming that port facilities are known to be so plentiful relative to shipping that the two restrictions under (1.4b) do not exclude any point in the commodity space attainable under the restriction (1.4a).

If, on the other hand, no explicit capacity limits are introduced, we shall still confine ourselves to situations in which supply of the services of port facilities is no problem. Since we are thus no longer interested in these primary commodities, we have in Table II omitted the port capacity rows of Table I. This leaves us with three variables of interest, y_{PQ}, y_{QP}, and $\bar{z}$. Our problem has thus been reduced to finding the possible point set in the space of these three commodity flows.

[4] Since, as is easily seen from Figure 1, Postulates B and D_2 of Chapter III, Section 3, are satisfied. See also Lemma 5.8.1 of the same chapter.

From Table II we obtain the following net output equations:

$$
\begin{array}{lrlllll}
(1.7a) & y_{PQ} = & & \bar{x}_{PQ}, & & & \\
(1.7b) & y_{QP} = & & & & \bar{x}_{QP}, & \\
(1.7c) & 0 = & - & \bar{x}_{PQ} - & x_{PQ} + & \bar{x}_{QP} + & x_{QP}, \\
(1.7d) & 0 = & & \bar{x}_{PQ} + & x_{PQ} - & \bar{x}_{QP} - & x_{QP}, \\
(1.7e) & \bar{z} = & & -t_{PQ}\bar{x}_{PQ} - & s_{PQ}x_{PQ} - & t_{QP}\bar{x}_{QP} - & s_{QP}x_{QP}.
\end{array}
$$

Since (1.7c) implies (1.7d), we can omit (1.7d) from consideration.[5]

1.3. *Leg-of-voyage and round-voyage activities.* At this point we must decide on a choice of coordinates in the activity space. One possibility is to operate in terms of the levels $\bar{x}_{PQ}$, x_{PQ}, $\bar{x}_{QP}$, x_{QP} of the "elementary" activities so far introduced while observing the restriction (1.7c). Another possibility is to complete the reduction of the technology matrix by introducing new "composite" activities, chosen in such a way that a combination of elementary activities at the levels $\bar{x}_{PQ}$, x_{PQ}, $\bar{x}_{QP}$, x_{QP} will at the same time be a combination of the new activities (at nonnegative levels) *if and only if* the restriction (1.7c) is satisfied. We can then operate with nonnegative combinations of the new activities without further restraints.

When $\bar{a}_{PQ}$, a_{PQ}, $\bar{a}_{QP}$, a_{QP} are written for the column vectors of coefficients in the technology matrix A of Table II, such a new set of activities is defined by

$$
(1.8) \quad \begin{array}{c} a_{(1)} = \bar{a}_{PQ} + \bar{a}_{QP}, \qquad a_{(2)} = \bar{a}_{PQ} + a_{QP}, \qquad a_{(3)} = a_{PQ} + \bar{a}_{QP}, \\ a_{(4)} = a_{PQ} + a_{QP}. \end{array}
$$

These activities represent the four different types of round voyages described in Table III. The unit of each of these activities is a rate of flow of one ship per month on the round voyage in question.[6]

[5] The two restrictions (1.7c) and (1.7d) are equivalent because we have a closed model with no activities that introduce or remove ships: one intermediate commodity row in Table II is the negative of the other. For that reason, for any constant levels of the activities, net appearances at port P are equal to the negative of net appearances at port Q. Since, in addition to requiring constant levels of all activities, our model classifies empty ship appearances as intermediate commodities, and thus prohibits accumulation (or decumulation) of ship inventories at any port, net appearances at both ports are required to be zero, and the two conditions expressing this are dependent.

[6] The use of round-voyage coordinates instead of leg-of-voyage coordinates corresponds to the use of loop currents instead of branch currents in the analysis of electrical networks. See Electric Circuits [1943], pp. 124–133. In Section 2.11 below we discuss the analogy with electrical networks further.

TABLE III. COMPLETELY REDUCED TECHNOLOGY MATRIX $\bar{A}$ FOR A TWO-PORT MODEL

Commodities			Activities *			
			Transporting cargo both ways	Transporting cargo P to Q, returning empty	Sailing empty P to Q, returning with cargo	Sailing empty both ways
	Unit	Symbol	x_1	x_2	x_3	x_4
Final:						
Cargo from P to Q	Shiploads per month	y_{PQ}	1	1		
Cargo from Q to P		y_{QP}	1		1	
Primary:						
Shipping	Ships	$\bar{\bar{z}}$	$-t_{PQ} - t_{QP}$	$-t_{PQ} - s_{QP}$	$-s_{PQ} - t_{QP}$	$-s_{PQ} - s_{QP}$

* In units of ships per month.

Since each new activity, regarded as a combination of the elementary "leg-of-voyage" activities, satisfies (1.7c), any combination of the new activities does. It follows further from (1.8) that any combination of the new activities with nonnegative levels, x_1, x_2, x_3, x_4, say, is a combination of the old activities with the nonnegative levels given by

$$\bar{x}_{PQ} = x_1 + x_2, \qquad x_{PQ} = x_3 + x_4, \qquad \bar{x}_{QP} = x_1 + x_3, \qquad x_{QP} = x_2 + x_4. \tag{1.9}$$

Conversely, for each set of nonnegative levels of the leg-of-voyage activities satisfying (1.7c) we can select nonnegative levels, x_1, x_2, x_3, x_4, of the round-voyage activities satisfying (1.9). One choice is given by

$$\begin{aligned} &x_1 = \bar{x}_{QP}, \quad x_2 = \bar{x}_{PQ} - \bar{x}_{QP}, \quad x_3 = 0, \quad x_4 = x_{PQ} \quad \text{if } \bar{x}_{PQ} \geqq \bar{x}_{QP}; \\ &x_1 = \bar{x}_{PQ}, \quad x_2 = 0, \quad x_3 = \bar{x}_{QP} - \bar{x}_{PQ}, \quad x_4 = x_{QP} \quad \text{if } \bar{x}_{QP} \geqq \bar{x}_{PQ}. \end{aligned} \tag{1.10}$$

In general, additional choices of the levels of the round-voyage activities can be derived from (1.10) by the transformation

$$x_1^* = x_1 + \delta, \qquad x_2^* = x_2 - \delta,$$
(1.10a)
$$x_3^* = x_3 - \delta, \qquad x_4^* = x_4 + \delta,$$

with such values of $\delta \neq 0$, if any, that the levels $x_1^*, \cdots$, remain nonnegative. The transformation (1.10a) consists in a trivial reshuffling of leg-of-voyage activities between ships in making up complete round voyages.

The vectors $a_{(1)}, \cdots$, defining the round-voyage activities, constitute the completely reduced technology matrix, $\bar{A}$, shown in Table III. They are frame vectors of the cone of all possible points in the space of the commodities y_{PQ}, y_{QP}, $\bar{z}$, as shown in Figure 1. (The term "possible" is used here in the meaning given to it in Chapter III, Section 4.1, i.e., possible without regard to availability limits on shipping or port capacities but under the restriction that ships circulate in a stationary flow pattern.)

In the case of n ports the number of round-voyage activities, $a_1, \cdots$, becomes large more quickly, with increasing n, than the number of leg-of-voyage activities, $\bar{a}_{PQ}, \cdots$. Accordingly, the degree of indeterminacy in their levels, $x_1, \cdots$, for given possible commodity flows, $y_{PQ}, \cdots$, $\bar{z}$, increases. Nevertheless it would seem that, for the purpose of specifying the entire possible point set, and its entire subset of efficient points (as defined below), the round-voyage coordinates, $x_1, \cdots$, are the proper ones to use. It would then be necessary to select those round-voyage activities, $a_{(k)}$, associated with frame vectors and, from these, select the subsets that can occur simultaneously in an efficient activity combination.

We have found leg-of-voyage coordinates, $\bar{x}_{PQ}, \cdots$, more useful, in the n-port case, to treat the more limited problem of deriving local properties of the efficient point set (i.e., properties in the neighborhood of one efficient point, as defined below). For this reason, in preparation for the n-port case, we shall in the present two-port case demonstrate the analysis of the efficient point set in terms of leg-of-voyage coordinates. It may be added that in dynamic models round-voyage coordinates lose their usefulness, whereas leg-of-voyage coordinates, properly dated, remain appropriate.

1.4. *Possible points and efficient points in the commodity space.* Equations (1.7a–e), in which the activity levels, $\bar{x}_{PQ}, \cdots$, are restricted to nonnegative values, define the set of (technologically) *possible* points in the space of the commodity flows (y_{PQ}, y_{QP}, $\bar{z}$). It follows from the analysis of Section 1.3 that this set is a convex polyhedral cone (A) spanned by the vectors $a_{(1)}$, $a_{(2)}$, $a_{(3)}$, $a_{(4)}$, as shown in Figure 1. In preparation for the analysis of the n-port model, we shall obtain the same result in leg-of-voyage coordinates.

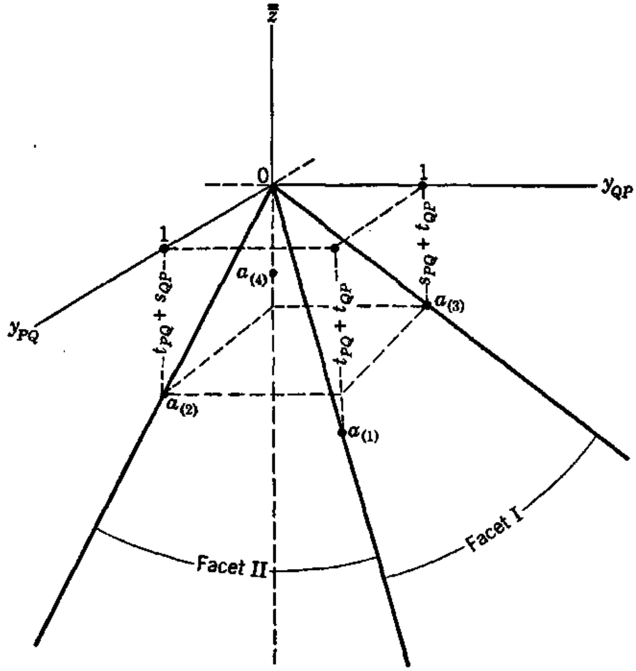


FIGURE 1—The possible cone in a two-port model. (Note: $\overline{0a}_{(4)} = s_{PQ} + s_{QP}$.)

For the purposes of the present analysis, a possible point $a = (y_{PQ}, y_{QP}, \bar{z}_0)$ is called *efficient* (without reference to the availability limit ζ on shipping) if $-\bar{z}_0$ represents the minimum amount of shipping required to carry out the program $y \equiv (y_{PQ}, y_{QP})$, i.e., if there exists no possible point $(y_{PQ}, y_{QP}, \bar{z})$ with $-\bar{z} < -\bar{z}_0$. This is an asymmetric definition, logically different from the definition adopted in Chapter III, Section 4.2. According to the latter definition, a point $a = (y_{PQ}, y_{QP}, \bar{z}_0)$ would be called efficient if there existed no possible point $a^* = (y^*_{PQ}, y^*_{QP}, \bar{z}^*)$ such that

$$(1.11) \qquad y^*_{PQ} \geqq y_{PQ}, \qquad y^*_{QP} \geqq y_{QP}, \qquad \bar{z}^* \geqq \bar{z}_0,$$

except the point $a^* = a$ itself. This says that, starting from an efficient point, no commodity flow can be increased (algebraically) without decreasing another commodity flow.

We shall show below that these two definitions, logically different, are equivalent under the assumptions of the present model. For that reason we can afford to utilize the former, asymmetric, definition in analyzing the efficient point set in Figure 1. For reasons of exposition we shall present this analysis independently of the general theory [III], referring to the theorems from which our results follow only after they have been obtained.

We shall therefore address ourselves to the following question: Given a program, $y = (y_{PQ}, y_{QP})$, what is the maximum value, $\bar{z} = \bar{z}_0$, of the nonpositive variable $\bar{z}$ such that $(y, \bar{z}) = (y_{PQ}, y_{QP}, \bar{z})$ is a possible point? The answer to this question will also help us to delineate the possible point set without using round-voyage coordinates.

We simplify the equations (1.7a–e) that define the possible point set, by eliminating $\bar{x}_{PQ}$ and $\bar{x}_{QP}$. This leaves us with the system

$$\text{(1.12a)} \qquad 0 = -y_{PQ} - x_{PQ} + y_{QP} + x_{QP},$$

$$\text{(1.12b)} \qquad -\bar{z} = t_{PQ}y_{PQ} + s_{PQ}x_{PQ} + t_{QP}y_{QP} + s_{QP}x_{QP},$$

$$\text{(1.12c)} \qquad y_{PQ} \geqq 0, \quad y_{QP} \geqq 0, \quad x_{PQ} \geqq 0, \quad x_{QP} \geqq 0,$$

where it is known that

$$\text{(1.13)} \qquad t_{PQ} > s_{PQ} > 0, \quad t_{QP} > s_{QP} > 0,$$

because of the definitions of the quantities involved.

From the given program $y = (y_{PQ}, y_{QP})$ we obtain

$$\text{(1.14)} \qquad x_{PQ} - x_{QP} = y_{QP} - y_{PQ} = \text{a given number}.$$

Starting with any set of values x_{PQ}, x_{QP} satisfying (1.14), we see readily that (1.14) is preserved if we diminish both x_{PQ} and x_{QP} by the same amount.[7] Hence, in view of (1.13), $\bar{z}$ is maximized by diminishing x_{PQ} and x_{QP} by the largest amount that does not violate (1.12c). We distinguish two cases.

CASE I: $y_{QP} - y_{PQ} \geqq 0$. Then efficiency requires

$$\text{(1.15)} \qquad x_{QP} = 0, \quad \text{and hence} \quad x_{PQ} = y_{QP} - y_{PQ} \geqq 0.$$

Substituting this in (1.12b) we obtain, with reference also to (1.12c),

$$\text{(1.16)} \qquad \begin{aligned} -\bar{z}_0 &= (t_{PQ} - s_{PQ})y_{PQ} + (t_{QP} + s_{PQ})y_{QP} \\ &\text{when} \quad y_{QP} - y_{PQ} \geqq 0, \quad y_{PQ} \geqq 0, \end{aligned}$$

a set of conditions defining what in anticipation we shall call facet I of the efficient point set.

[7] This diminution is a simple case of what will be called a circular transformation in Section 2.4.

The economic meaning of the foregoing reasoning is obvious and trivial. If the traffic with cargo from Q to P exceeds that in the opposite direction, the most efficient routing of ships in a continuing program requires the excess of shipping arising at P to be moved back in ballast to fill the deficit at Q, with no other movements in ballast taking place ($x_{QP} = 0$). Moreover, since it is always possible under (1.14) to waste tonnage by adding equal positive amounts to x_{PQ} and x_{QP}, we find that in Case I all points $(y_{PQ}, y_{QP}, \bar{z})$ satisfying

$$-\bar{z} \geqq (t_{PQ} - s_{PQ})y_{PQ} + (t_{QP} + s_{PQ})y_{QP}, \quad y_{QP} - y_{PQ} \geqq 0, \quad y_{PQ} \geqq 0 \tag{1.17}$$

are possible points.

CASE II: $y_{QP} - y_{PQ} \leqq 0$. Now efficiency requires

$$x_{PQ} = 0, \qquad \text{and hence} \qquad x_{QP} = y_{PQ} - y_{QP} \geqq 0. \tag{1.18}$$

We must send a number of empty ships from Q to P just sufficient to permit carrying out the transportation program. Sending empty ships from P to Q would, obviously, be inefficient. Substituting in (1.12b), we obtain

$$-\bar{z}_0 = (t_{PQ} + s_{QP})y_{PQ} + (t_{QP} - s_{QP})y_{QP} \quad \text{when} \quad y_{QP} - y_{PQ} \leqq 0, \quad y_{QP} \geqq 0, \tag{1.19}$$

which defines what we shall call facet II of the efficient point set. Moreover, all points $(y_{PQ}, y_{QP}, \bar{z})$ satisfying

$$-\bar{z} \geqq (t_{PQ} + s_{QP})y_{PQ} + (t_{QP} - s_{QP})y_{QP}, \quad y_{PQ} \leqq 0, \quad y_{QP} \geqq 0 \tag{1.20}$$

are possible.

We shall now demonstrate how the conditions (1.17) and (1.20) enable us to visualize the efficient facets I and II in relation to the entire possible point set. At the same time we shall show that facets I and II together constitute the entire efficient point set, also by the symmetric definition (1.11).

We note that the Cases I and II considered above exhaust all possibilities for the two nonnegative variables y_{PQ} and y_{QP}. Since (1.16) and (1.19) represent minimum values of $-\bar{z}$ for given values of y_{PQ}, y_{QP}, it follows that all possible points satisfy either (1.17) or (1.20), and some satisfy both conditions. Moreover, in Case I, (1.17) implies the first condition (1.20) because then

$$\begin{aligned}(t_{PQ} - s_{PQ})y_{PQ} + (t_{QP} + s_{PQ})y_{QP} - (t_{PQ} + s_{QP})y_{PQ} \\ - (t_{QP} - s_{PQ})y_{QP} = (s_{PQ} + s_{QP})(-y_{PQ} + y_{QP}) \geqq 0.\end{aligned} \tag{1.21}$$

Similarly, in Case II, (1.20) implies the first condition (1.17). It follows that the possible point set is fully described by the following four inequalities:

$$y_{PQ} \geqq 0, \qquad y_{QP} \geqq 0,$$

$$(1.22) \qquad 0 \geqq (t_{PQ} - s_{PQ})y_{PQ} + (t_{QP} + s_{PQ})y_{QP} + \mathfrak{z},$$

$$0 \geqq (t_{PQ} + s_{QP})y_{PQ} + (t_{QP} - s_{QP})y_{QP} + \mathfrak{z}.$$

The possible point set is therefore an intersection of halfspaces, each having the origin in its boundary, hence a convex polyhedral cone. Since possible points can be found in the boundary of each of these halfspaces, the cone as shown in Figure 1 has four two-dimensional facets, of which two fall in coordinate planes. The remaining two facets, I and II, constitute the set of efficient points by either definition, asymmetric or symmetric, because by (1.13) all coefficients in the last two inequalities (1.22) are positive. Hence, if (1.16) or (1.19) holds in a point $a = (y_{PQ}, y_{QP}, \mathfrak{z}_0)$ an increase in any one or more of the coordinates of a destroys the possibility of that point. On the other hand, any possible point a not satisfying (1.16) or (1.19) permits some increase to its $\mathfrak{z}$-coordinate, say, without destroying its possibility.

We note that the present analysis in leg-of-voyage coordinates has led us to a characterization of the possible cone as an intersection of halfspaces, while the analysis of Section 1.3 in round-voyage coordinates has led us to an equivalent characterization as a convex hull of halflines, $(a_{(1)}), \cdots, (a_{(4)})$. This equivalence is discussed more generally by Gale [XVII].

1.5. *The efficient point set as a transformation function.* The conditions for efficiency can be summarized in the statement that it is inefficient for ballast traffic in both directions to be positive. Therefore

$$(1.23) \qquad \text{either} \quad x_{PQ} = 0 \quad \text{or} \quad x_{QP} = 0.$$

Thus, facet I of the efficient point set [less the halfline $(a_{(1)})$ common to both facets] may be characterized as the set of efficient points for which $x_{PQ} > 0$, and hence $x_{QP} = 0$. Similarly, facet II [less the halfline $(a_{(1)})$] can be characterized as the set of efficient points for which $x_{QP} > 0$, and hence $x_{PQ} = 0$. Thus, to anticipate a bit, an efficient facet may be identified with the empty shipping route(s) in use for all points on the relative interior [III, Section 2.4; XVIII, Definition 31] of that facet. We also observe that the efficient point set in Figure 1 represents a transformation function (production function) defined everywhere in the range of its variables and expressed, for instance, as the minimum shipping requirements,

$$-\bar{z}_0 = \min(-\bar{z}) = f(y_{PQ}, y_{QP}), \tag{1.24}$$

of a given program. The derivatives of this function and the ratio thereof,

$$\frac{\partial \bar{z}_0}{\partial y_{PQ}}, \quad \frac{\partial \bar{z}_0}{\partial y_{QP}}, \quad \frac{\partial \bar{z}_0}{\partial y_{QP}} \Big/ \frac{\partial \bar{z}_0}{\partial y_{PQ}}, \tag{1.25}$$

where they exist, represent the marginal cost, in terms of shipping employed, of a unit increase in the program items y_{PQ} and y_{QP}, respectively, and the marginal rate of substitution expressing the opportunity cost of a unit increase in y_{QP} in terms of a compensating decrease in y_{PQ}. On facet I the marginal rates of substitution (1.25) take the values

$$t_{PQ} - s_{PQ}, \qquad t_{QP} + s_{PQ}, \qquad (t_{QP} + s_{PQ})/(t_{PQ} - s_{PQ}). \tag{1.26}$$

Thus the marginal cost of transporting an additional shipload per month from Q to P on facet I, in the proper units, is given by

$$\frac{\Delta \bar{z}_0}{\Delta y_{QP}} = \frac{(t_{QP} + s_{PQ}) \text{ ships}}{1 \text{ ship per month}} = (t_{QP} + s_{PQ}) \text{ months}, \tag{1.27}$$

where Δ denotes corresponding finite increments in the variable following it. This marginal cost coefficient equals the full turn-around time of a ship returning empty because on facet I the preponderant cargo movement is in the direction y_{QP} (i.e., the direction in which loaded traffic is being increased). No return loads are available for the additional ships, and the time cost of their return trip must be charged to the increment in outgoing cargo movements.

On the other hand, the cost of transporting an additional shipload monthly from P to Q on facet I is

$$\frac{\Delta \bar{z}_0}{\Delta y_{PQ}} = \frac{(t_{PQ} - s_{PQ}) \text{ ships}}{1 \text{ ship per month}} = (t_{PQ} - s_{PQ}) \text{ months}. \tag{1.28}$$

This is the time cost of reallocating a ship from sailing in ballast from P to Q to transporting cargo from P to Q. We may regard t_{PQ} in (1.28) or t_{QP} in (1.27) as the *direct cost* of the additional transportation commitment, occasioned by the operations with cargo, and $-s_{PQ}$ in (1.28) or s_{PQ} in (1.27) as its *indirect cost*, occasioned by the change in location of the ship resulting from its loaded movement.

Similarly, on facet II, the rates of substitution (1.24) are

$$t_{PQ} + s_{QP}, \qquad t_{QP} - s_{QP}, \qquad (t_{QP} - s_{QP})/(t_{PQ} + s_{PQ}). \tag{1.29}$$

The marginal rates of substitution, (1.26) and (1.29), are applicable both to *finite increases* and *finite decreases* in cargo flows, *within the facet in* *question.* In particular, the marginal cost coefficient for an increase in the preponderant cargo flow (y_{QP} in Case I) applies to indefinitely large increases as long as we rule out port congestion. The marginal cost coefficient for a decrease in the lesser cargo flow (y_{PQ} in Case I) applies until that flow is reduced to zero and is therefore restricted in its applicability only by the *feasibility limit* ($\Delta y_{PQ} \geqq -y_{PQ}$ in Case I) to the decrease in question. The marginal cost coefficient for an increase in the lesser flow, and for a decrease in the preponderant flow, are subject to *applicability limits* that are reached at the relative boundary [III, Section 2.4; XIX, Definition 32] of the facet in question, where the two cargo flows have become equal. From a point, $y_{PQ} = y_{QP}$, on the halfline $(a_{(1)})$ common to facets I and II, the marginal cost of a unit increase in one of the cargo flows exceeds the marginal saving from a unit decrease in that flow.

1.6. *Efficiency prices.* Although the foregoing analysis is complete for the two-port model, it may be useful to indicate an equivalent characterization of the efficient point set which will be helpful in analyzing the n-port model. The coefficients of y_{PQ}, y_{QP}, $\bar{z}_0$ in equation (1.16) of facet I are the coordinates

$$(1.30) \qquad p_{PQ} = t_{PQ} - s_{PQ}, \qquad p_{QP} = t_{QP} + s_{PQ}, \qquad p_z = 1,$$

of a vector p normal to that facet. They have been interpreted as efficiency prices [III, Sections 4.7, 5.12] associated with each efficient point of facet I. It has been proved generally for linear models of production such as those considered here [III, Theorems 4.3, 5.11] that a possible point a in the commodity space is efficient if and only if there exists an associated price vector p (subject to certain sign restrictions on its components) such that each activity in the technology has a nonpositive profitability, while each activity engaged in, in order to realize the commodity flow vector $a = \sum_k a_{(k)}x_k$, has a zero profitability

$$(1.31) \quad \begin{cases} \text{(a)} & p'a_{(k)} \leqq 0 \quad \text{for all } k, \\ \text{(b)} & p'a_{(k)} = 0 \quad \text{if} \quad x_k > 0. \end{cases}$$

The sign restrictions relevant to the present case are that p shall have positive components for all desired (final or primary) commodities, i.e., all commodities entering in the definition of efficiency, and represented in our case by the flows y_{PQ}, y_{QP}, $\bar{z}$:

$$(1.32) \qquad p_{PQ} > 0, \qquad p_{QP} > 0, \qquad p_z > 0.$$

No sign restriction is involved for the prices of intermediate commodities, here represented by the flows y_P and y_Q. In the presence of intermediate commodities with net flows restricted to zero, the criterion stated as applied to the original technology is equivalent to that applied to the reduced technology from which intermediate commodities have been eliminated [III, Theorem 5.11].

In the present example, it is most easily verified from the reduced matrix in Table III that conditions (1.31) and (1.32) indeed admit all points, a, of facets I and II, and no other points. The vector p of *efficiency prices* defines a halfspace, $p'a^* \leqq 0$, containing all possible points, a^*, and having the efficient point a in its bounding plane. This bounding plane necessarily contains the facet having a in its relative interior.[8] If normalized by

$$p_z = 1, \tag{1.33}$$

the vector p is therefore uniquely determined, at the same value (1.30), for all points, a, of the relative interior of facet I. Similarly it uniquely equals

$$p_{PQ} = t_{PQ} + s_{QP}, \qquad p_{QP} = t_{QP} - s_{QP}, \qquad p_z = 1 \tag{1.34}$$

on all relative interior points, a, of facet II. It is not uniquely determined at any point a of the common relative boundary of two two-dimensional facets, but may be given the value specific to either facet, or a positive linear combination of these two values [subject to the sign restriction (2.32) which enters when one of the two facets is not an efficient facet]. Thus the efficiency prices define marginal rates of substitution wherever they are uniquely determined.

1.7. *The efficiency price on the location of ship appearances.* The equivalent application of criteria (1.31) and (1.32) to the technology matrix of Table II leads to the determination of efficiency prices on the intermediate commodities (ship appearances) which will play an important role in the n-port model. On the relative interior of facet I, where we have

$$\bar{x}_{PQ} > 0, \qquad x_{PQ} > 0, \qquad \bar{x}_{QP} > 0, \qquad x_{QP} = 0, \tag{1.35}$$

the conditions (1.31) now become

$$\begin{aligned} p_{PQ} - p_P + p_Q - t_{PQ} &= 0, \\ -p_P + p_Q - s_{PQ} &= 0, \\ p_{QP} + p_P - p_Q - t_{QP} &= 0, \\ p_P - p_Q - s_{QP} &\leqq 0. \end{aligned} \tag{1.36}$$

[8] The definition of relative interior referred to above implies that the facet having a in its relative interior is the facet spanned by those vectors $a_{(k)}$ for which $x_k > 0$ in (1.31). See Gerstenhaber [XVIII, Theorem 1].

These conditions are solved, within the sign restrictions (1.32) and subject to the normalization (1.33), by (1.30) and by any p_P, p_Q such that

$$p_Q = p_P + s_{PQ}. \tag{1.37}$$

We note that the efficiency prices, p_P and p_Q, of ship appearances in P and Q permit the following general expressions for the marginal cost coefficient for an increase in a cargo flow. From (1.26) and (1.37) we derive

$$\frac{\Delta \bar{z}_0}{\Delta y_{PQ}} = t_{PQ} + p_P - p_Q, \qquad \frac{\Delta \bar{z}_0}{\Delta y_{QP}} = t_{QP} + p_Q - p_P \tag{1.38}$$

for finite variations within facet I. Since these expressions are symmetrical in the two ports, they apply also to finite variations within facet II if we remember that on that facet we must replace (1.37) by

$$p_P = p_Q + s_{QP}. \tag{1.39}$$

By (1.38) the indirect cost of an increase in a transportation commitment by one shipload a month, accomplishable within any one efficient facet, is found to equal the decrease in the efficiency price attached to the location of a ship, resulting from the change in location required by the fulfilment of the new transportation commitment. For that reason the prices p_P and p_Q have also been called the *economic potential* of the location of a ship in P and Q, respectively.[9]

1.8. *Efficient points under capacity restrictions.* In the foregoing analysis the input $-\bar{z}$ of shipping has been regarded as a variable entering into the definition (1.11) of efficiency. Alternatively, in the definition of efficiency, the third condition (1.11) can be replaced by the requirement that both the would-be efficient point a and the possible point a^* be attainable, i.e., satisfy the capacity restriction (1.4a) arising from a given size of fleet. In this case the attainable point set is the pyramid $0A_1A_2A_3A_4$ indicated by Figure 2a, and the efficient point set consists of the two line segments A_1A_2 and A_1A_3. Figure 2b gives the corresponding attainable ($0A_1A_2A_3$) and efficient (A_1A_2 and A_1A_3) point sets in the two-dimensional "program space" of the variables y_{PQ}, y_{QP}.

The foregoing analysis of efficiency prices remains valid in the present case, provided that the assumption that the limits to port capacities are never reached is maintained. The only difference is that efficiency now requires that the quantity $\bar{z}_0$, previously a freely choosable nonpositive

[9] The reader may wish to exercise himself in the application of this concept to models with three or four ports.

variable, now equals the given constant ζ. Therefore the interpretation of efficiency prices, when unique, as marginal rates of substitution in efficient operation now applies only to offsetting variations in y_{PQ} and y_{QP}. Further interpretations of the efficiency prices are given in the n-port case in Sections 2.8, 2.9, and 2.10.

If port capacities ζ_P, ζ_Q actually restrict the attainable point set by (1.4b), the foregoing analysis of efficiency prices applies only in those efficient points (if any) in which neither of the equality signs in (1.4b)

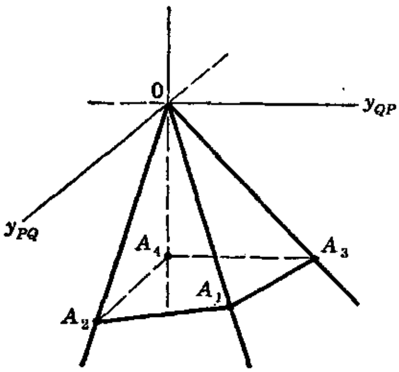


FIGURE 2a

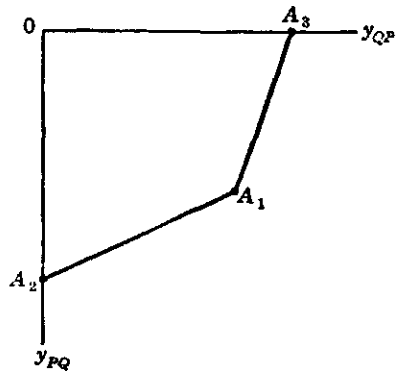


FIGURE 2b

applies. If one or both equality signs in (1.4b) apply in an efficient point, the efficiency prices associated with such a point, if unique, contain allowances for rent arising from the use of scarce port facilities.

2. A MODEL WITH n PORTS

2.1. *The routing of empty ships.* A generalization to n ports of the technology matrix of Table II is given in Table IV. This table contains a cargo transportation activity and an empty sailing activity for each route (i.e., for each ordered pair of ports). The units of measurement of activities and commodities have been changed to correspond to an example described in Section 2.2 below.

As in the two-port case, we begin by assuming a desired transportation program. Let us treat this program as if it were given to a central shipping authority whose job it is to perform the indicated transportation, unchanged from month to month, at minimum cost in terms of shipping continually in use.

The given transportation program will determine the levels of all activities relating to the movement of cargo. This leaves the shipping authority free to choose only the levels of activities involving movements

TABLE IV. PARTIALLY REDUCED TECHNOLOGY MATRIX FOR AN n-PORT MODEL

Commodities			Activities *													
			Transporting cargo								Sailing empty					
			from *1* to *2*	from *1* to *3*	...	from *1* to n	from *2* to *1*	from *2* to *3*	...	from n to $n-1$	from *1* to *2*	from *1* to *3*	...	from *1* to n	...	from n to $n-1$
	Unit	Symbol	$\bar{x}_{12}$	$\bar{x}_{13}$	...	$\bar{x}_{1n}$	$\bar{x}_{21}$	$\bar{x}_{23}$	...	$\bar{x}_{n,(n-1)}$	x_{12}	x_{13}	...	x_{1n}	...	$x_{n,(n-1)}$
Final:																
Cargo transported																
From *1* to *2*	1 million	y_{12}	1													
From *1* to *3*	tons of	y_{13}		1												
...	cargo per	...														
From *1* to n	month	y_{1n}				1										
From *2* to *1*		y_{21}					1									
From *2* to *3*		y_{23}						1								
...		...														
From n to $n-1$		$y_{n,(n-1)}$								1						
Intermediate:																
Net appearances of empty ships																
At *1*	1 million	0	−1	−1		−1	1				−1	−1		−1		
At *2*	tons of	0	1				−1	−1			1					
At *3*	cargo-	0		1				1				1				
...	carrying	...														
At $n-1$	capacity	0								1						1
At n	per month	0				1				−1				1		−1
Primary:																
Shipping	1 million tons of cargo-carrying capacity	z	$-t_{12}$	$-t_{13}$		$-t_{1n}$				$-t_{n,(n-1)}$	$-s_{12}$	$-s_{13}$				$-s_{n,(n-1)}$

* In millions of tons of cargo or cargo carrying capacity per month.

of empty ships. Thus the problem of finding the efficient point corresponding to a given transportation program is equivalent to finding a routing plan for empty ships such that the total cost of the given program, in terms of shipping in use, is minimized.

These statements, obvious in themselves, are verified if we write the net output equations associated with the technology matrix of Table IV in the matrix form [10]

$$a \equiv \begin{bmatrix} y \\ 0 \\ \bar{z} \end{bmatrix} = \begin{bmatrix} I & 0 \\ B & B \\ -t' & -s' \end{bmatrix} \begin{bmatrix} \bar{x} \\ x \end{bmatrix} \equiv A\bar{\bar{x}} \qquad (y \geqq 0, \quad \bar{\bar{x}} \geqq 0). \tag{2.1}$$

From the first set of rows we find the loaded movements $\bar{x}$ determined by the program y,

$$y = I\bar{x} = \bar{x}. \tag{2.2}$$

We further read from the last (single) row

$$\bar{\bar{z}} = \bar{z} + z, \quad \text{with} \quad \bar{z} \equiv -t'\bar{x} = -t'y, \quad z \equiv -s'x, \tag{2.3}$$

where, analogous to (1.13), we have

$$t > s > 0. \tag{2.4}$$

We thus find that, in order to minimize shipping employed, $-\bar{\bar{z}}$, in a given program, y, we must minimize shipping employed in empty movements, $-z$, by proper choice of the routing plan x so as to balance the loaded movements $\bar{x}$. To achieve this balancing x must satisfy the restrictions

$$Bx + b = 0, \quad \text{with} \quad b \equiv B\bar{x}, \tag{2.5}$$

following from the second set of rows in (2.1).

The matrix B, a submatrix of the given technology matrix A, is found by reference to Table IV. To visualize the conditions (2.5), it will be useful to write them equivalently in indicial form,

$$\begin{aligned} &x_{ij} \geqq 0, \qquad \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = b_i, \\ &\text{where} \quad b_i \equiv \sum_{j=1}^{n} \bar{x}_{ji} - \sum_{j=1}^{n} \bar{x}_{ij}, \qquad (i = 1, \cdots, n). \end{aligned} \tag{2.6}$$

(It is convenient here to think of x_{ii}, $\bar{x}_{ii}$ as equal to zero for all i, and not represented in the vectors x, $\bar{x}$.) The elements of b defined in (2.6) are

[10] We employ the notations for inequalities between vectors introduced in Chapter III, Section 2.5.

the net surpluses of empty ships arising in the various ports from the performance of loaded movements. These are actual surpluses in all ports receiving more shiploads than they dispatch, deficits in all ports in the reverse situation. Because the net surpluses are generated by loaded movements, and because our model of shipping technology disregards ship losses at sea, the sum of all net surpluses vanishes, as is easily verified directly from (2.6). If

$$e' \equiv [1 \quad 1 \quad \cdots \quad 1] \tag{2.7}$$

is a row vector with all its n elements equal to 1, the remark just made is expressed by the property

$$e'B = 0 \tag{2.8}$$

of the matrix[11] B, which through (2.5) leads to

$$e'b = \sum_i b_i = 0. \tag{2.9}$$

From the nature of our problem it is intuitively obvious that for every program y there exists at least one routing plan x that minimizes shipping employed in empty movements. To argue this point mathematically, we note first that every program y is possible (i.e., can be performed if enough shipping $-\hat{z}$ is available and port capacities are unlimited). A particular routing plan which is always available[12] is

$$x_{ij} = \bar{x}_{ji}, \qquad \text{so} \quad -z = \sum_{i,j} s_{ij}\bar{x}_{ji} = \sum_{i,j} s_{ij}y_{ji} \equiv -z_0, \text{ say.} \tag{2.10}$$

This routing plan consists in returning all ships empty by the routes reverse to those traveled with cargo. Although in general this routing plan will be inefficient, it permits us to remark that, if we do not have, for all $i, j, i \neq j$,

$$x_{ij} \leqq -z_0/s_{ij}, \quad \text{where}^{13} \quad 0 < -z_0/s_{ij} < \infty, \tag{2.11}$$

the function $-z = s'x$ will exceed the value $-z_0$ reached in (2.10). For the purpose of finding the minimum of the linear function $s'x$ subject to the restrictions (2.5), therefore, we can confine x to the closed and bounded convex polyhedral set S given by (2.5) and (2.11). We state without proof that on such a set a linear function reaches a minimum.

[11] It may be noted that B is of rank $n - 1$ and can by permutation of columns be given the form $[-C \quad C]$. It follows that b is subject to no other restrictions than (2.9), even though $\bar{x}$ is restricted to be nonnegative.

[12] Another type of "feasible solution" is employed by Dantzig [XXIII] as initial value in an iterative method to find an optimal x.

[13] Using (2.4) and excluding the trivial case $y = 0$, $z_0 = 0$, where $x = 0$ minimizes $-z$.

TABLE V. NET RECEIPTS OF DRY CARGO IN OVERSEAS TRADE, 1913 *

Area	Representative Port	Received Annually, $12\sum_j y_{ji}$	Dispatched Annually, $12\sum_j y_{ij}$	Annual Net Receipts, $12b_i$	Monthly Average of Annual Net Receipts, b_i
Baltic countries, Norway, Germany, Netherlands, Belgium, Great Britain, and Ireland	Rotterdam	151.30	156.72	− 5.42	−0.46
France, Spain, and Portugal	Lisbon	38.48	23.63	14.85	1.24
Mediterranean, except France, Spain, and Portugal	Athens	32.42	13.86	18.56	1.55
Black Sea countries	Odessa	1.70	13.25	−11.55	−0.96
West Africa	Lagos	2.76	1.34	1.42	0.12
South and East Africa	Durban	2.93	1.77	1.16	0.10
Arabia, Iran, and India	Bombay	6.49	9.95	− 3.46	−0.29
Malaya, Siam, Indochina, Philippines, and Indonesia	Singapore	4.75	4.93	− 0.18	−0.02
Japan, China, and Asiatic Russia	Yokohama	5.39	3.35	2.04	0.17
Australia and New Zealand	Sydney	3.37	6.30	− 2.93	−0.24
Pacific Coast of United States and Canada	San Francisco	2.60	2.37	0.23	0.02
Atlantic and Gulf Coast of United States and Canada	New York	12.78	28.18	−15.40	−1.28
Mexico, Caribbean, North Coast of South America, and Brazil	St. Thomas	12.04	7.80	4.24	0.35
Remainder of South America	La Plata	12.26	15.82	− 3.56	−0.30
Total		289.27	289.27	0.00	0.00

* Source: *Der Güterverkehr der Weltschiffahrt*, Statistisches Reichsamt, Berlin, 1928. All figures are in millions of metric tons.

This minimum may be reached in a single point, x, or in all points of a closed and convex polyhedral set, $S_{\min}$. No local minima higher than this absolute minimum can exist because of the convexity of S. For proofs of these statements we refer to Chapter III, Sections 5.6, 5.8.

2.2. *An example with data for* 1913. We shall employ an example [14] constructed from data showing world movements of dry cargo in 1913. Table V gives the computation of the net surplus vector b from the given program. Although the data represent shipments between areas, we shall assume for simplicity that the entire traffic of an area goes through its representative port. The sailing times, s_{ij}, defined on that basis can be derived from Table VI. Assuming that a ship in ballast (allowing

TABLE VI. DISTANCES BETWEEN REPRESENTATIVE PORTS *

	Lisbon	Athens	Lagos	Durban	Yokohama	San Francisco	St. Thomas
Rotterdam	1.1						
Odessa		0.7					
Bombay	5.2	3.7	8.1	4.1	5.4	9.8	8.3
Singapore	7.2	5.7	9.0	4.9	2.9	7.3	10.2
Sydney	10.6	9.0	12.7	6.2	4.3	6.4	8.8
New York	3.0	4.8	4.9		9.7	5.2	1.4
La Plata	5.3	7.1	4.3	4.6	13.2	8.7	4.6

* All figures are in 1,000 nautical miles. Assuming vessel speeds in ballast (with allowance for time spent fueling) to correspond to 5,000 nautical miles per month, the coefficients s_{ij} are found by dividing the figures in this table by 5.

for time spent in refueling) sails 5,000 nautical miles in one month, the coefficients s_{ij} are obtained by dividing the figures in Table VI by 5. Furthermore, while the data were actually generated in a market situation, we shall assume them to be given to our hypothetical central shipping authority as the desired transportation program. Finally, we proceed as if these numbers represented constant rates of flow through time.

The unit of cargo flows is a million tons per month. The unit of flows of shipping, loaded or empty, is the flow of ships that if loaded would carry a unit flow of cargo. In this choice of units, we have disregarded the slight dependence of a ship's carrying capacity on the length of the loaded voyage, arising from the necessity to carry fuel.

[14] A similar example based on data for 1925 is contained in Koopmans [1947].

2.3. *Possible graphs of empty shipping routes.* Any vector x satisfying (2.6), and hence representing a possible routing plan for empty ships in relation to the program y, defines a set of routes (i, j) on which a positive flow of empty ships, $x_{ij} > 0$, is prescribed. The figure consisting of all these routes (as "arcs") plus all ports in the technology (as "vertices") is, according to topological terminology, a *linear graph* [König, 1936]. It will be called the graph $G = G(x)$ of ballast traffic associated with the possible routing plan x.

Map 1 gives a possible graph of ballast traffic for the program of Table V. Amounts x_{ij} that satisfy condition (2.6) are indicated alongside each route, the net surpluses b_i with each port.

2.4. *Conditions for an efficient graph.* A routing plan, x, possible in relation to a program, y, is called efficient if it minimizes the amount (2.4) of shipping absorbed in empty movements. A graph, $G(x)$, associated with an efficient routing plan, x, is likewise called efficient. A general theorem [III, Theorem 4.3, as extended in Theorem 5.9], already quoted, states that the existence of a vector p of prices, satisfying conditions similar to (1.36) as well as certain sign restrictions, is necessary and sufficient for the efficiency of a point (2.1) and, hence, of a routing plan x in relation to a program y. We shall here establish the validity of this criterion by reasoning specific to the present transportation model. The shortest road to that end will be the heuristic exploration of the consequences of conditions (1.36), should they be satisfied at a point a arising by (2.1) from a routing plan x.

In the present notation, and normalizing by $p_z = 1$, the conditions (1.36) are, for cargo-carrying activities,

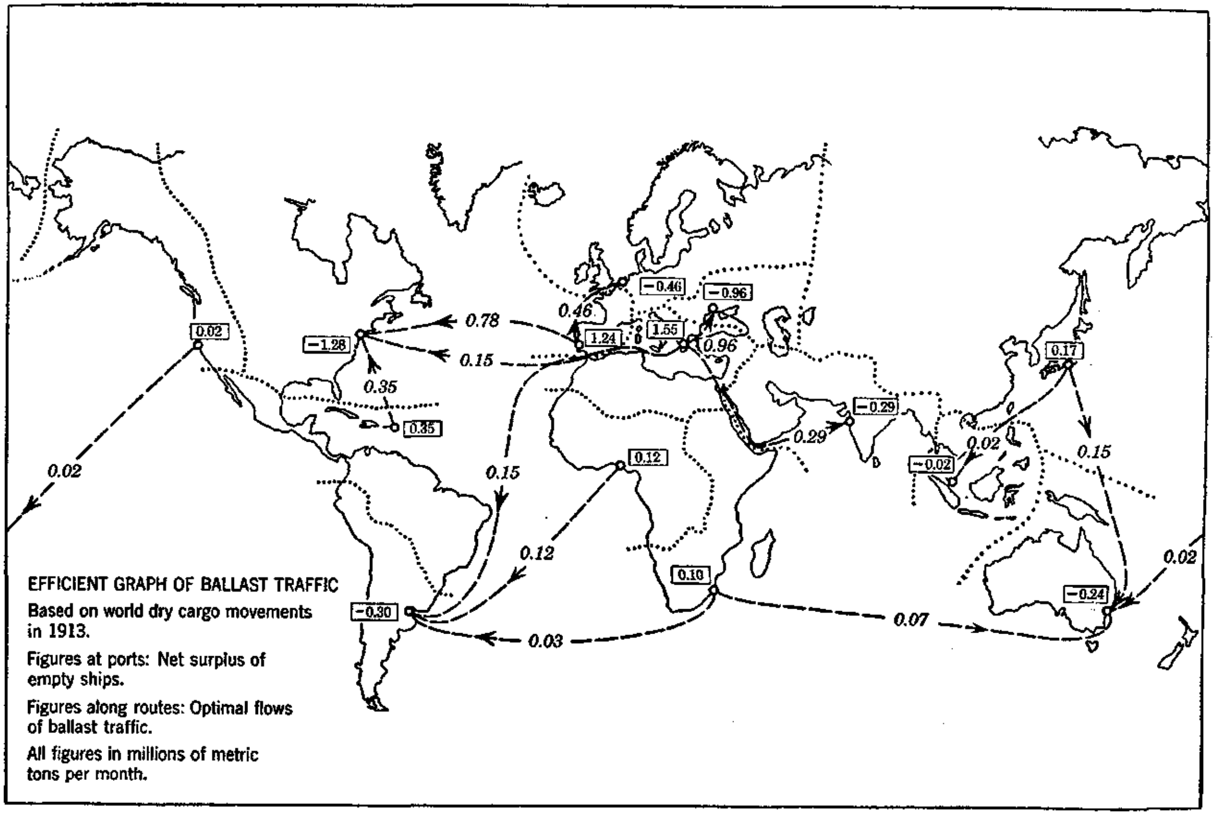
$$(2.12)\quad \begin{cases} \text{(a)} \qquad p_{ij} - p_i + p_j - t_{ij} \leqq 0 \qquad \text{for all } (i, j), \\ \text{(b)} \qquad p_{ij} - p_i + p_j - t_{ij} = 0 \qquad \text{if} \quad \bar{x}_{ij} > 0, \end{cases}$$

and, for empty movements,

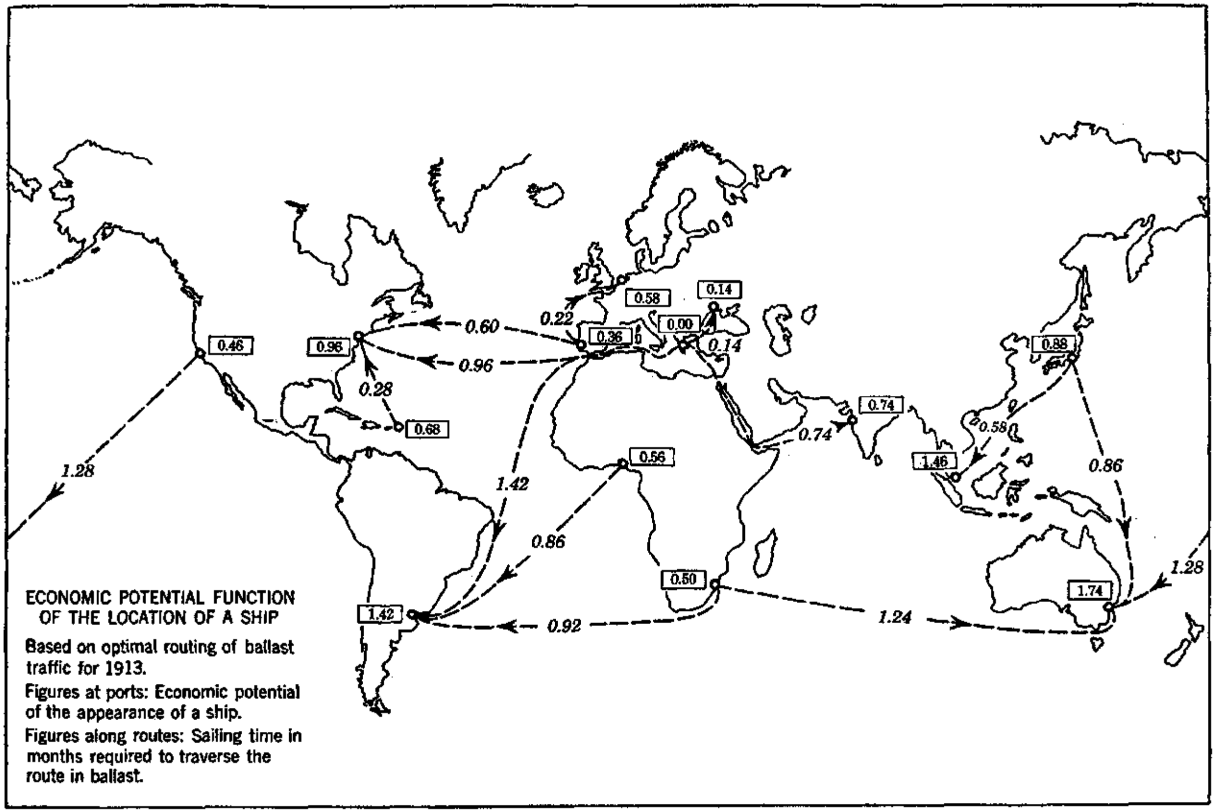
$$(2.13)\quad \begin{cases} \text{(a)} \qquad p_j \leqq p_i + s_{ij} \qquad \text{for all } (i, j), \\ \text{(b)} \qquad p_j = p_i + s_{ij} \qquad \text{if} \quad x_{ij} > 0. \end{cases}$$

We shall concern ourselves first with the conditions (2.13) on the p_i, which, as prices of intermediate commodities, are not subject to any sign restrictions. In Map 2 we shall attempt to determine for all ports values p_i that satisfy the conditions (2.13) corresponding to the graph G_{1913} of Map 1. Since conditions (2.13) are invariant under addition of the same constant to all p_i we may arbitrarily choose

$$(2.14)\qquad p_{\text{Athens}} = 0.$$


EFFICIENT GRAPH OF BALLAST TRAFFIC
Based on world dry cargo movements in 1913.
Figures at ports: Net surplus of empty ships.
Figures along routes: Optimal flows of ballast traffic.
All figures in millions of metric tons per month.
0.02
−1.28
0.78
0.15
0.35
0.35
0.46
−0.46
−0.96
1.24
1.55
0.96
0.17
−0.29
0.29
0.02
−0.02
0.15
0.12
0.12
0.15
0.02
0.02
0.10
−0.24
−0.30
0.03
0.07

MAP 1


ECONOMIC POTENTIAL FUNCTION
OF THE LOCATION OF A SHIP
Based on optimal routing of ballast traffic for 1913.
Figures at ports: Economic potential of the appearance of a ship.
Figures along routes: Sailing time in months required to traverse the route in ballast.
0.46
0.96
0.68
0.58
0.14
0.00
0.36
0.56
0.50
1.42
0.74
1.46
0.88
1.74
0.60
0.96
0.28
0.22
0.14
1.28
1.42
0.86
0.92
1.24
0.74
0.58
0.86
1.28

MAP 2

Application of (2.13b), with reference to Table VI for the values of s_{ij}, now leads successively to the values of p_i in all ports connected with Athens by a chain of routes,

$$
\begin{aligned}
p_{\text{LaPlata}} &= p_{\text{Athens}} + 1.42 = 1.42, \\
p_{\text{Durban}} &= p_{\text{LaPlata}} - 0.92 = 0.50,
\end{aligned}
\tag{2.15}
$$

etc., as exhibited in Map 2. The procedure is formalized as follows.

We shall exclude graphs which contain both a route (l, m) and its reverse (m, l). Such graphs would in any case be inefficient. Let a chain C contained in the graph G be defined as a sequence of routes of G,

$$(i, j) \text{ or } (j, i), \cdots, (l, m) \text{ or } (m, l), \cdots, (p, q) \text{ or } (q, p), \tag{2.16}$$

connecting successive ports in a sequence,

$$i, j, \cdots, l, m, \cdots, p, q, \tag{2.17}$$

of ports, no two of which are the same. This chain is said to lead from port i to port q. With reference to the routes of a given chain, C, we define [15]

$$
\begin{cases}
\nu_{lm}^{C} = 1 & \text{if } (l, m) \in C \text{ and } l \text{ precedes } m \text{ in (2.17)}, \\
\nu_{ml}^{C} = -1 & \text{if } (m, l) \in C \text{ and } m \text{ follows } l \text{ in (2.17)}.
\end{cases}
\tag{2.18}
$$

Then repeated application of (2.13b) is equivalent to the rule

$$p_q = p_i + \sum_{(g, h) \in C} \nu_{gh}^{C} s_{gh}, \qquad C \subset G, \qquad C \text{ leads from } i \text{ to } q. \tag{2.19}$$

In order to ascertain whether and when this procedure gives determinate p_i-values in all ports, we must explore the two possibilities of contradiction and of indeterminacy. To begin with possible contradiction, let us define a circuit, O, contained in G in a manner similar to a chain, except that we require the first and last port to be the same $(q = i)$, all other ports to be different. The notion of a circuit includes the sense (from left to right) in which the sequence (2.16) (with $q = i$) is traced, and constants ν_{gh}^{O} are defined again by (2.18) with reference to that sense.

In a graph that contains no circuits, (2.19) cannot lead to a contradiction. If a graph G contains a circuit O with successive ports $(i, j, k, \cdots, p, i)$, contradiction will arise unless

$$\sum_{(g, h) \in O} \nu_{gh}^{O} s_{gh} = 0. \tag{2.20}$$

This is easily seen to be also a necessary condition for the efficiency of G.

[15] The symbol $\in$ denotes "is a route of"; the symbol $\subset$, "is contained in."

If the left-hand member of (2.20) were negative, a *circular transformation*,

$$(2.21)\qquad \begin{aligned} x^*_{gh} &= x_{gh} + \mu\nu^O_{gh} \quad \text{if} \quad (g, h) \in O, \\ x^*_{gh} &= x_{gh} \text{ on all other routes } (g, h), \end{aligned}$$

of the flows of empty shipping, with a positive *modulus* μ, would decrease the expression (2.4) for the shipping engaged in these movements by the positive amount

$$(2.22)\qquad s'x - s'x^* = -\mu \sum_{(g,h)\in O} \nu^O_{gh} s_{gh}.$$

The meaning of this transformation is that, tracing the circuit in the sense $i, j, k, \cdots, p, i$, an amount μ is added to the flow of empty ships on all routes traced in the direction of that flow and an amount μ is subtracted from the flow on all routes traced in a direction opposite to that flow. This can always be done within the restriction $x^* \geqq 0$ by taking a sufficiently small value of μ, since the graph $G = G(x)$ contains only routes of positive flows $x_{ij}, \cdots$. Similarly, if the left-hand member of (2.20) were positive, a sufficiently small negative modulus μ would define a possible transformation such that (2.22) is positive. Finally, by considering each of the four cases (omitting superscripts O)

$$(2.23)\qquad \begin{gathered} \nu_{ij} = \nu_{jk} = 1; \qquad \nu_{ji} = \nu_{kj} = -1; \qquad \nu_{ij} = 1, \quad \nu_{kj} = -1; \\ \nu_{ji} = -1, \quad \nu_{jk} = 1; \end{gathered}$$

it is easily seen that, with either sign of μ, the transformed routing plan x^* satisfies the restrictions (2.6) for every port j whenever the original plan x does.

A circuit $O = (i, j, \cdots, i)$ contained in a graph G is called *neutral* if (2.20) is satisfied. It is easily seen that, if all circuits contained in a graph G are neutral, no contradiction can arise in defining p_i-values in all ports. Since, in particular, any circuit contained in an efficient graph is a neutral circuit, no contradiction in the evaluation of p_i-values for all ports by (2.19) can arise if the graph G is indeed efficient.

Indeterminacy of one or more p_i can arise if the graph G is not *connected*, i.e., if there exists at least one pair of ports i, q not connected by a chain $(i, j, \cdots, q)$ in G. We shall come back to this case in Section 2.5, and we assume here that G is indeed connected.

We have thus established that, if G is a possible and connected graph containing only neutral circuits (if any), conditions (2.13b) permit a unique determination of p_i-values for all ports from a prescribed value in one port. We shall prove further that, under these assumptions, conditions (2.13a) are both necessary and sufficient for the efficiency of G.

To show the necessity, assume that

$$p_j > p_i + s_{ij} \tag{2.24}$$

for some (i, j). To illustrate the argument in Map 2, let i = Durban, j = Singapore, and assume that $s_{ij} = 0.92$ instead of the value 0.98 following from Table VI. Then, by adding the route (Durban, Singapore) to G_{1913} to make G^*_{1913}, we give rise to a circuit (Durban, Singapore, Yokohama, Sydney, Durban) in G^*_{1913} because Singapore and Durban are already connected by a chain in G before the new route is added. The saving in shipping from a circular transformation on this circuit with positive modulus μ equals, by (2.22) and (2.19),

$$\begin{aligned} &-\mu(s_{\text{Du, Si}} - s_{\text{Yo, Si}} + s_{\text{Yo, Sy}} - s_{\text{Sy, Du}}) \\ &\qquad\qquad = -\mu(s_{\text{Du, Si}} + p_{\text{Du}} - p_{\text{Si}}), \end{aligned} \tag{2.25}$$

which, by assumption (2.24), is a positive number. Hence the graph G_{1913} is not efficient because savings can be secured by sending empty ships on a route outside it. It should be added that in the reverse case,

$$p_j < p_i + s_{ij}, \tag{2.26}$$

no saving can be effected by a negative choice of μ because $x_{ij} = 0$ before the transformation, and any negative value of μ would make $x^*_{ij} < 0$, which is technologically impossible.

To show the sufficiency, assume that (2.13a) holds for the p_i determined from (2.13b). Define a row vector,

$$p'_B \equiv [p_1 \quad p_2 \quad \cdots \quad p_n], \tag{2.27}$$

to contain all p_i. We can then write (2.13a), which was obtained by applying (1.31a) to the second set of columns in the technology matrix A as partitioned in (2.1), in the form

$$p'_B B - s' \leqq 0. \tag{2.28}$$

In particular, from (2.13b), i.e., from (1.31b) applied to A, the equality sign in (2.28) applies to all components such that the corresponding component of x is positive. Since the remaining components of x are zero, we have

$$(p'_B B - s')x = 0. \tag{2.29}$$

Now let x^0 be any possible routing plan, i.e., any vector $x^0 \geqq 0$ satisfying (2.5) if x^0 is substituted for x, giving

$$Bx^0 = -b = Bx. \tag{2.30}$$

Then, since $x^0 \geqq 0$, we have from (2.28)

$$(2.31) \qquad (p_B'B - s')x^0 \leqq 0.$$

Comparison of (2.29) and (2.31), using (2.30) after premultiplication by p', leads to

$$(2.32) \qquad s'x^0 \geqq s'x.$$

Hence there is no possible routing plan x^0 employing less shipping than x, and G is an efficient graph.

We have thus found that, in the case of a connected graph G, the existence of a solution p_i of (2.13) is by itself a necessary and sufficient condition for the efficiency of G.

This does not conflict with the theorem mentioned at the beginning of this section, which includes (2.12) in the condition. For, if p_i satisfies (2.13), the p_{ij} defined by requiring equality in (2.12) for all routes (i, j), whether cargo is moved on them or not, will be positive as a consequence of the inequalities

$$(2.33) \qquad t_{ij} > s_{ij} \qquad \text{for all } (i, j)$$

similar to (1.13).

A comparison of the p_i-values in Map 2 with the s_{ij} as derived from Table VI establishes that the graph of that map is indeed efficient.

2.5. *The case of a disconnected graph.* The proof that the existence of a solution p_i of (2.13) is sufficient for the efficiency of a graph G does not depend on G being connected. The proof of the necessity of that condition needs to be supplemented for the case of a disconnected graph G. A disconnected graph is possible only if the set of ports can be partitioned into two or more subsets such that the net surpluses b_i add up to zero within each subset. This, again, can only happen by "accident," by "special" choice of the program vector y. Since we have specified that all ports belong to any graph of ballast traffic, the case where a certain port is neither the origin nor the destination of any route of empty ships constitutes, and properly so, a special case of a disconnected graph.

As an illustration of a disconnected graph, add 0.9 million tons to the cargo flow from Durban to Sydney. The graph $G_{1913}^{\dagger}$ obtained from that of Map 1 by deleting the route (Durban, Sydney) is a possible and disconnected graph for the modified program $y^{\dagger}$ so obtained. Does the existence of a solution p_i of (2.13) remain a sufficient criterion for its efficiency? We shall sketch the reasoning that leads to an affirmative answer.

Let us refer to the two connected subgraphs of $G^{\dagger}_{1913}$ as the Atlantic and Pacific subgraphs. Values for the p_i in all "Atlantic" ports (with which we include Durban and Bombay) are uniquely determined, by (2.13b), from the value (2.14) arbitrarily assumed for Athens. Similarly, p_j-values for all "Pacific" ports are expressed by

$$p_j = \lambda + q_j, \qquad \lambda \equiv p_{\text{Yokohama}}, \tag{2.34}$$

say, and where the q_j are uniquely determined by (2.13b). Consider all routes leading from an "Atlantic" port i to a "Pacific" port j. For each such route we read from (2.13a) an inequality,

$$\lambda + q_j \leqq p_i + s_{ij}. \tag{2.35}$$

For each route of the reverse type we have, similarly,

$$\lambda + q_j \geqq p_i - s_{ji}. \tag{2.36}$$

Hence a value of λ which satisfies (2.13a) can be found only if

$$\max_{\substack{i \in \text{Atl} \\ j \in \text{Pac}}} (p_i - q_j - s_{ji}) \leqq \min_{\substack{i \in \text{Atl} \\ j \in \text{Pac}}} (p_i - q_j + s_{ij}). \tag{2.37}$$

If this is not true, there exist "Atlantic" ports i_1, i_2 and "Pacific" ports j_1, j_2 such that

$$p_{i_1} - q_{j_1} - s_{j_1 i_1} > p_{i_2} - q_{j_2} + s_{i_2 j_2}, \tag{2.38}$$

and a circuit can be found which contains the routes (j_1, i_1) and (i_2, j_2) and on which a circular transformation with positive modulus produces a saving in shipping employed.

This reasoning can be extended to cover the case of three or more connected subgraphs, not mutually connected, of a possible graph G.

2.6. *Iterative computation of an efficient graph.* The foregoing proof of the necessity of the conditions (2.13) for the efficiency of a graph G suggests a method of iterative improvement of a tentative possible initial graph, G_1. Such a method is described by Dantzig [XXIII]. The difference $p_j - p_i$ is there referred to as the indirect cost of the activity measured by x_{ij} and is denoted by $\bar{c}_{ij}$.

Table VII gives corresponding notations in Dantzig's chapter and the present one. In making the comparisons it should be kept in mind that the program of commodity inflows and outflows, assumed given for the various terminals by Dantzig, corresponds to our net surpluses of shipping at the various ports. His minimum cost transportation program of a homogeneous commodity corresponds to our efficient routing plan of empty ships.

TABLE VII. CORRESPONDING NOTATIONS IN CHAPTERS XXIII AND XIV

XXIII	XIV	XXIII	XIV
c_{ij}	s_{ij}	$a_1, \cdots, a_m, b_1, \cdots, b_n$	$b_1, \cdots, b_n$
x_{ij}	x_{ij}	B_k	a_{ij} such that $x_{ij} > 0$
A_{ij}	a_{ij}	x_k	x_{ij} such that $x_{ij} > 0$
z	$-z$	$\nu_k = \lambda_{lk} + \mu_{mk}$, if $B_k = A_{lm}$	ν_{lm}, if $x_{lm} > 0$
$\bar{c}_{ij}$	$p_j - p_i$	$\nu_k = \lambda_{mk} + \mu_{lk}$, if $B_k = A_{ml}$	ν_{ml}, if $x_{ml} > 0$
u_i	$-p_i$	$c_1, \cdots, c_{m+n-1}$	s_{ij} such that $x_{ij} > 0$
v_j	p_j		

2.7. *Routing plans associated with an efficient graph.* So far our attention has been directed to the graph G of ballast traffic (i.e., the set of routes for which $x_{ij} > 0$) rather than to the actual values of the x_{ij} and the resulting value $\bar{z}$ of shipping employed in ballast traffic. We shall now establish the connection between G and x.

Assume first that G contains no circuits and is connected, taking as an example the graph G_{1913} of Map 1 in relation to the unchanged program y_{1913} of Table V. Such a graph, known as a tree, uniquely determines the flows, x_{ij}, of empty ships on all its routes. To show the determination of the flow of 0.9 million of (empty) cargo-carrying capacity on the route (Durban, Sydney), delete that route from G_{1913}. This recreates the graph $G^{\dagger}_{1913}$ previously considered, but this is not a possible graph in relation to the program y_{1913}. Either of its two trees can be used to determine uniquely the value

$$x_{\text{Du, Sy}} = \sum_{i \in \text{Atl}} b_i = -\sum_{i \in \text{Pac}} b_i = 0.9 \tag{2.39}$$

from a summation of the relations (2.6) over all ports of one tree. If G is not connected, the same reasoning can be applied to each of its connected subgraphs.

Each circuit contained in a graph G introduces the possibility of a circular transformation in x, with a modulus limited by a lower and an upper possibility bound depending on the initial routing plan, x_0, say. It can be shown that the set of routing plans x compatible with such a graph G forms a convex polyhedron of a dimensionality equal to the *cyclomatic number* of G (i.e., the maximum number of routes that can be removed from G without disconnecting any pair of ports connected within G) [König, 1936].

If G is efficient, all its circuits are neutral circuits, and all routing plans x associated with G lead to the same value, $-z$, of shipping employed in ballast traffic, which is the minimum value of $-z$ among all possible routing plans.

If G is efficient and connected, then (2.13b) uniquely determines the p_i, and what we shall call the *maximal efficient graph* $\bar{G}$ is obtained by adding to G all routes (i, j) for which the equality sign in (2.13a) holds. (To obtain the maximal efficient graph $\bar{G}$ if G is not connected may require adding two or more routes simultaneously to avoid impossible graphs.) If $\bar{G}$ does not contain a circuit, the efficient routing plan x is unique.

Neutral circuits often occur in practice. Many would be present in the technology of Table VI if all its cells were filled out, but it so happened that none entered in the efficient graph of Map 1. Neutral circuits arise whenever all four routes connecting either of two ports i, j with either of two ports k, l go past the same geographical point (cape, narrow passage).

2.8. *The marginal cost of variations in the program.* We shall now show that the efficiency prices, p_{ij}, on cargo flows, when uniquely determined by (2.12) and (2.13), define marginal rates of substitution of cargo flows against shipping. These substitution rates are applicable to all changes in the program that can be balanced by a change in the efficient routing plan x without causing an essential change in the corresponding graph G of ballast traffic.

As an example, consider the addition of $\mu = 0.1$ million tons of cargo to the monthly flow from New York to San Francisco in the 1913 program of Table V. Since G_{1913} is connected, the route (NY, SF) can be supplemented by routes in G_{1913} to a circuit (NY $\rightarrow$ SF $\rightarrow$ Sy $\leftarrow$ Du $\rightarrow$ LP $\leftarrow$ Ath $\rightarrow$ NY), in which the arrows indicate the direction of ballast traffic on all but the first route. The change in the program can be effected, within the restrictions (2.6), by the circular transformation

$$\begin{aligned} &\bar{x}^*_{\mathrm{NY,SF}} = \bar{x}_{\mathrm{NY,SF}} + \mu, \qquad x^*_{\mathrm{SF,Sy}} = x_{\mathrm{SF,Sy}} + \mu, \\ (2.40) \quad &x^*_{\mathrm{Du,Sy}} = x_{\mathrm{Du,Sy}} - \mu, \qquad \cdots, \qquad x^*_{\mathrm{Ath,NY}} = x_{\mathrm{Ath,NY}} + \mu, \\ &x^*_{ij} = x_{ij} \text{ on all other routes.} \end{aligned}$$

Because of the moderate amount of its modulus, μ, the graph $G(x^*)$ after this transformation is the same as the original graph, G_{1913}. Since the efficiency of a possible routing plan depends only on the efficiency of its graph, it follows that x^* is again an efficient routing plan for the changed program.

The unit cost of the transformation (2.40) (i.e., the cost divided by the modulus, μ, expressed in terms of additional shipping used) is

$$(2.41) \qquad t_{\mathrm{NY, SF}} + s_{\mathrm{SF, Sy}} - s_{\mathrm{Du, Sy}} + \cdots + s_{\mathrm{Ath, NY}}.$$

By (2.19) and (2.18), this cost equals

$$(2.42) \qquad t_{\mathrm{NY, SF}} + p_{\mathrm{NY}} - p_{\mathrm{SF}}.$$

On the other hand, this expression equals the efficiency price $p_{\mathrm{NY, SF}}$ as determined by (2.12b) whenever $y_{\mathrm{NY, SF}} = \bar{x}_{\mathrm{NY, SF}} > 0$ in the original program, or as permitted by (2.12a) if $y_{\mathrm{NY, SF}} = 0$. This establishes the interpretation of the p_{ij}, when uniquely determined, as marginal cost coefficients,

$$(2.43) \qquad p_{ij} = t_{ij} + p_i - p_j.$$

The term t_{ij} in (2.43) can be called the direct cost of a unit addition to the program on the route (i, j), the term $p_i - p_j$ the indirect cost. The indirect cost arises because, on completion of its loaded movement, the ship is in a different location and hence has a different locational potential. The term allowing for this circumstance is the loss in potential (in the efficiency price of ship appearance) associated with the loaded movement.

Since all relationships involved are linear, the coefficients p_{ij}, if unique, can also be used to express the simultaneous cost of a number of program changes Δy_{ij},

$$(2.44) \qquad -\Delta \bar{z} = \sum_{i,j} p_{ij} \Delta y_{ij}.$$

The validity of this expression is limited to changes in the program which permit the same potential function p_i to apply before and after the change, with the p_{ij} defined by (2.12b) on all routes. This is certainly the case if the same efficient graph applies before and after the change. It remains true in certain boundary cases whenever the efficient graphs G and G^* before and after the change are contained in the same maximal efficient graph $\bar{G}$. It can be shown that all programs y permitting routing plans x whose graphs are contained in the same maximal efficient graph $\bar{G}$ form a closed facet [16] of the efficient point set. The maximum dimensionality of such a facet is $n(n - 1)$, the number of variables y_{ij}, $\bar{z}$, less one. This maximum is reached if $\bar{G}$ is connected, in which case the efficiency price vector p is uniquely determined in every point in the relative interior of the facet in question, and represents the normal to that facet.

[16] For the concept of a facet, see Chapters III, Section 4.5, and XVIII, Section 4.

The cost expression (2.44) for a change in the program y within a facet can be decomposed into direct and indirect cost as follows. If the p_{ij} are to be unique, conditions (2.12b) must apply to all routes (i, j). If we write p_y for the vector with elements p_{ij} ordered as the y_{ij} in Table IV, the normalized efficiency price vector p is given by

$$p' = [p_y' \quad p_B' \quad 1]. \tag{2.45}$$

In this notation, with reference to the technology matrix A as partitioned in (2.1), conditions (2.12b) for all routes can be written as

$$[p_y' \quad p_B' \quad 1]\begin{bmatrix} I \\ B \\ -t' \end{bmatrix} = p_y' + p_B'B - t' = 0, \tag{2.46}$$

and hence, from (2.41),

$$-\Delta\bar{z} = t'\Delta y - p_B'B\Delta y. \tag{2.47}$$

In this expression, in view of (2.2) and (2.5),

$$B\Delta y = B\Delta\bar{x} = \Delta b \tag{2.48}$$

represents the vector of changes in net shipping surpluses b_i in the various ports, resulting from the change Δy in the program. Substituting (2.48) in (2.47), we find that the cost,

$$-\Delta\bar{z} = t'\Delta y - p_B'\Delta b = \sum_{i,j} t_{ij}\Delta y_{ij} - \sum_i p_i \Delta b_i, \tag{2.49}$$

of a change in the program within an $n(n-1)$-dimensional closed facet of the efficient point set is the sum of a direct cost, $\sum t_{ij}\Delta y_{ij}$, representing the net increase in shipping employed in cargo-transporting activities, and an indirect cost, $-\sum p_i\Delta b_i$, representing the net increase in shipping efficiently employed in empty movements. The latter cost can be obtained, without tracing the changes Δx_{ij} in flows of empty ships on individual routes, as the negative of the sum of the changes Δb_i in the net shipping surpluses, each multiplied by the economic potential, p_i, of a ship in the location of that surplus.[17]

2.9. *Uses of the efficiency prices by a central shipping authority.* In a pool of shipping administered by a central authority, such as existed in the first and second world wars, an efficient routing plan x and a set of efficiency prices p corresponding to a program y can become known to that authority only by explicit computations based on the performance

[17] Because of (2.9), the expression (2.49) is not affected by the addition of a constant to the potential function p_i.

times t_{ij}, s_{ij}. Once computed, the prices p_{ij} can be used to assess the opportunity cost of the acceptance of one transportation commitment, in terms of other commitments that have to be rejected, if the total amount of shipping available for active operations [18] is limited. Where two possible commitments are not competitive but substitutes, such as when the same raw material can be obtained from two different sources of supply or when the location of a raw material processing activity is to be selected, calculations based on the prices p_{ij} are needed to arrive at the best solution. It should be emphasized again that these uses are subject to all the limitations of the present analysis. They apply to the comparison of alternative programs, each constant over time and both compatible with the same efficient graph $G(x)$. They therefore do not analyze the cost of transition in time from one constant program to another. These and many other problems in the centralized operations of a pool of shipping can only be approached by dynamic generalizations of the foregoing theory.

2.10. *Efficiency prices as market prices under competition.* The conditions (2.12) and (2.13) would also be fulfilled, with proper interpretation, by freight rates p_{ij} formed in a competitive market in which the composition of demand for transportation services on the various routes is stable and in which shipowners independently bid for and carry out transportation commitments, whenever necessary moving their ships empty to a more advantageous position for the fulfilment of the next commitment. Of course, market freight rates also contain allowances for the cost of other scarce factors besides shipping, represented by such things as port and canal dues, allowance for accumulating repair needs, wages of crews and stevedores, cost of fuel and supplies. To keep matters simple, let us assume that all factors other than the use of shipping can be bought *at constant prices* in any desired quantities. Then we may add one row to the technology matrix, stating for each route the money cost of the amounts of these factors required for the unit of each activity. The commodity flow corresponding to this new row is money input, and the efficiency price can conveniently be taken equal to unity so as to express all other efficiency prices in money terms.

In a model for long-run analysis it would be appropriate to treat the cost of the use of shipping in the same manner by making an overhead charge, including depreciation and interest, based on the money cost of ship construction. However, since adjustment in the size of the world merchant shipping fleet is much slower than the fluctuations in demand-

[18] That is, the total fleet in existence less an allowance for ships in repair or overhauling.

at-constant-price for its services, it is also of interest, in an analysis which is neither too long-run nor too short-run in character, to consider temporarily constant demand schedules on all routes in conjunction with a temporarily constant size of the fleet, which is not necessarily in long-run equilibrium with these demand schedules.

In this case, the row in the technology matrix corresponding to input $\hat{z}$ of shipping should be retained and the efficiency price p_z on the use of shipping be interpreted as a "rental charge" for the use of one unit of shipping during one unit of time. This charge is expressed by the market as a "time-charter rate," at which the use of a ship is traded.[19] The time-charter rate expresses the "scarcity" of ships in the period in question, in terms of benefits forgone, or cost incurred by alternative methods, because there is not one more ship available. When in a depression ships are laid up idle, the time-charter rate as here understood is zero.[20]

So interpreted, conditions (2.12) and (2.13) express that the profit on any round voyage that is actually engaged in, or that can be pieced together from legs-of-voyage (with or without cargo) actually engaged in, is zero to the entrepreneur, provided that he calculates the time-charter charge as a cost. The profit is positive on no round voyage, and negative on inefficient round voyages. This is indeed the result of entrepreneurs' decisions in a perfectly competitive market, according to accepted static equilibrium theory. If market demand, y_{ij}, for transportation services on the various routes remains constant for a sufficiently long time, the efficiency prices, p_{ij}, are observable as freight rates per shipload on the various routes. The economic potential function, p_i, is implicit in the calculations of the shipowners in choosing between alternative round voyages. The type of contract that would make the p_i observable as market prices has to our knowledge not been in use in ocean shipping or in any other transportation market.

Where competition is restricted, such as in line shipping, discrepancies between freight rates and efficiency prices may result. This is even more true, empirically, in transportation systems subject to government operation or regulation. This is not an inevitable consequence of governmental activity, but rather of the simple and crude notions of "fairness" which have historically dominated such activity under the watching eyes of highly interested local and functional groups of population and industry. The resulting inefficiency in the geographical dis-

[19] The type of time charter approximating most closely the concept of a time-charter rate here applicable is known as the "bareboat charter," by which the use of a ship is handed over for a period without crews or supplies.

[20] This statement is still based on the (unrealistic) assumption, made in Section 1, that all ships are of the same type and quality.

tribution of industry has been briefly commented on elsewhere [Koopmans, 1947].

2.11. *Analogy with Kirchhoff's law on the distribution of current in an electrical network.* There is an interesting analogy, with differences, between the problem of minimizing the amount of shipping in use for a given transportation program and the distribution of (direct) current in a network of electrical conductors to which given electromotive forces are applied at specified points. The latter problem, treated by Kirchhoff [1847], provided the stimulus for the mathematical investigation of linear graphs.[21] The analogy is brought out by the following list of reinterpretations of the symbols used above.

Interpretation in Transportation Model	*Symbol*	*Interpretation in Electrical Network*
Ports	$i = 1, \cdots, n$	Connection points of conductors
Routes	(i, j)	Conductors
Empty sailing time	s_{ij}	Resistance
Flow of empty ships	x_{ij}	Electrical current
Net shipping surplus	b_i	Net current made to flow into the network from outside
Locational potential	p_i	Negative of electrical potential

In the electrical application the identity (2.6) expresses that the total inflow of current into a connection point of conductors must equal the total outflow. Kirchhoff's law on the determination of the currents x_{ij} in the various conductors can be derived from the minimization of the total heat,

$$h = \kappa \sum_{i,j} s_{ij} x_{ij}^2, \tag{2.50}$$

generated per unit of time in the network, subject to the restraints (2.6) on the currents. This differs decisively from the transportation problem, in which, instead of the quadratic form (2.50), the linear form

$$-z = \sum_{i,j} s_{ij} x_{ij} \tag{2.51}$$

is minimized subject to the additional restriction $x_{ij} \geqq 0$. The heat minimization problem lends itself naturally to application of calculus

[21] The cultural lag of economic thought in the application of mathematical methods is strikingly illustrated by the fact that linear graphs are making their entrance into transportation theory just about a century after they were first studied in relation to electrical networks, although organized transportation systems are much older than the study of electricity.

by the method of Lagrange parameters. Because the function (2.50) is a sum of squares with positive coefficients, placed under linear restraints, the minimizing solution x_{ij} is unique. It is found to be such that, instead of (2.13b),

$$s_{ij}x_{ij} = -p_i + p_j, \tag{2.52}$$

where $-p_i$ is the electrical potential at the connection point i, obtained as a Lagrange parameter associated with the corresponding restraint (2.6).

It is possible to apply the same method to the minimization of shipping in use, by the substitution

$$x_{ij} = w_{ij}^2, \qquad w_{ij} \text{ real}, \tag{2.53}$$

which insures that the condition $x_{ij} \geqq 0$ is met. The locational potential then plays the same role of a set of Lagrange parameters. The difference in sign in its definition is motivated by the fact that a ship is more useful at the destination of an empty voyage than at the point of origin, whereas an electric particle may be regarded as more usefully located where the electrical potential is high.

Because the degree in the x_{ij} of (2.51) is one less than that of (2.50), the difference in potential between two points i, j in the transportation model is related to the "resistance" or "time cost" only in the case that $x_{ij} > 0$, and is within that case independent of the value of x_{ij}. Also, the minimizing solution x_{ij} lacks uniqueness if neutral circuits occur in the maximal efficient graph.