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Perspectives

# Graph factors and factorization: 1985–2003: A survey

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Dedicated to the memory of Peter Owens

**Abstract**

In the most general sense, a *factor* of a graph $G$ is just a spanning subgraph of $G$ and a graph *factorization* of $G$ is a partition of the edges of $G$ into factors. However, as we shall see in the present paper, even this extremely general definition does not capture all the factor and factorization problems that have been studied in graph theory. Several previous survey papers have been written on this subject [F. Chung, R. Graham, Recent results in graph decompositions, London Mathematical Society, Lecture Note Series, vol. 52, Cambridge University Press, 1981, pp. 103–123; J. Akiyama, M. Kano, Factors and factorizations of graphs—a survey, J. Graph Theory 9 (1985) 1–42; L. Volkmann, Regular graphs, regular factors, and the impact of Petersen's theorems, Jahresber. Deutsch. Math.-Verein. 97 (1995) 19–42] as well as an entire book on graph decompositions [J. Bosák, Decompositions of Graphs, Kluwer Academic Publishers Group, Dordrecht, 1990]. Our purpose here is to concentrate primarily on surveying the developments of the last 15–20 years in this exponentially growing field.


*Keywords:* Factor; Factorization; Matching

## 1. Introduction

A subgraph $H$ of a graph (multigraph, general graph) $G$, is a *factor* of $G$ if $H$ is a spanning subgraph of $G$. This definition is sufficiently general so as to include such prominent problems in graph theory as edge coloring and Hamilton cycles to name but two. Although both of these problems will be mentioned in our survey, we shall concentrate on other areas of graph factors, leaving the reader to consult existing surveys on these two topics (cf. [108,109,126,127]).

We shall take as our departure point an excellent survey of graph factors up to 1985 written by Akiyama and Kano [12].

Hence we shall concentrate on works published since their paper appeared, although we shall certainly mention some of the basic results on factors obtained prior to the Akiyama–Kano paper, but which form the basis of more recent work.

Akiyama and Kano divided the field of graph factors into two classes of problems. They named these two classes *degree* factor problems and *component* factor problems and we shall follow their lead in this.

A factor $F$ described in terms of its degrees will be called a *degree* factor. For example, if a factor $F$ has all of its degrees equal to 1, it is called a 1-*factor* (or a *perfect matching*). On the other hand, if the factor is described via some other graphical concept, it is called a *component* factor. For example, if each component of the factor $F$ is a tree, $F$ is

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a component factor. We hasten to observe that these two classes are not disjoint. For example, finding a 1-factor (i.e., a factor in which each vertex has degree one) and finding a factor each component of which is an edge amounts to the same thing.

If the edge set of a graph $G$ can be represented as the edge-disjoint union of factors $F_1, F_2, \ldots, F_k$, we shall refer to $\{F_1, F_2, \ldots, F_k\}$ as a *factorization* of graph $G$.

There is a vast body of work on factors and factorizations and this topic has much in common with other areas of study in graph theory. For example, factorization significantly overlaps the topic of *edge coloring*. Indeed, any color class of a proper edge coloring of a graph is just a matching. Moreover, the *Hamilton cycle problem* can be viewed as the search for a connected factor in which the degree of each vertex is exactly two.

We will treat factors of *finite undirected* graphs only. However, a number of papers dealing with infinite graph factors and directed graph factors are included in our reference list. We do not discuss these results here, but instead refer the interested reader to [317,2–8,49,168,268–270,313,334–337,283,284].

Let us now present some of the basic definitions, notation and terminology used in this paper. Other terminology will be introduced as it naturally occurs in the text. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. In this paper, all graphs will be considered *simple*, unless otherwise specified. That is to say, there is at most one edge joining any pair of vertices. By *multigraph* we mean that there may be multiple edges joining the same pair of vertices. If graph $G$ admits a vertex partition $V(G) = V_1 \cup \cdots \cup V_j$ such that every edge of $G$ joins two different $V_i$'s, we say that $G$ is *multipartite*. If $j = 2$, we say that $G$ is *bipartite*. The *complete bipartite graph* $K_{m,n}$ has $m$ vertices in one class, $n$ in the other and all pairs of vertices from different classes are joined by an edge.

A *path* in graph $G$ is an alternating sequence of distinct vertices and edges beginning and ending with vertices in which each edge joins the vertex before it to the one following it. The *length* of a path $P$ is the number of edges in $P$. $P_i$ will designate a path containing exactly $i$ vertices. A graph $G$ is *connected* if every pair of vertices in $G$ are joined by a path in $G$. Otherwise, it is *disconnected*. If every pair of vertices are joined by an edge, we say that the graph is *complete* and if, in addition, $|V(G)| = n$, we denote this graph by $K_n$.

Graph $H$ is a *subgraph* of graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of $G$ *spans* $G$ if $V(H) = V(G)$. A subgraph $H$ of $G$ is *induced* if every pair of vertices in $H$ which are adjacent in $G$ are also adjacent in $H$. An induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$ is called a *claw*. A graph containing no $K_{1,3}$ as an induced subgraph is said to be *claw-free*. More generally, if $H$ and $G$ are graphs and $G$ does not contain $H$ as an induced subgraph, we shall say that $G$ is *H-free*. A *component* of $G$ is a maximal connected subgraph of $G$. A set of vertices $S \subseteq V(G)$ is a *cutset* of a connected graph $G$ if $G - S$ is disconnected. The cardinality of any smallest cutset in $G$ is called the *connectivity* of $G$ and is denoted by $\kappa(G)$. (As usual, we define $\kappa(G) = 0$ if $G$ is disconnected and $\kappa(K_n) = n - 1$ for the complete graph $K_n$.) Similarly, the cardinality of any smallest set of edges in $G$ the removal of which disconnects the graph is called the *edge-connectivity* of $G$ and is denoted by $\lambda(G)$. If $e$ is an edge in a connected graph $G$ such that $G - e$ is disconnected, we say that $e$ is a *bridge* or *cutedge*.

The *degree* of a vertex $v$, denoted $\deg_G(v)$, or simply $\deg(v)$, when the underlying graph is understood, is the number of edges incident with the vertex. The minimum degree in graph $G$ will be denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. A sequence $\{d_1, \ldots, d_n\}$ of non-negative integers is said to be *graphical* if there is a graph $G$ the vertex set of which can be labelled so as to have $\deg_G(v_i) = d_i$, for $i = 1, \ldots, n$. A graph is *r-regular* if the degree of each vertex in $G$ is $r$ and the graph is *regular* if it is $r$-regular for some $r$. A set of vertices is *independent* if no two of its members are joined by an edge. The cardinality of any largest independent set of vertices in $G$ is called the *independence number* of $G$ and is denoted by $\alpha(G)$. A *cycle* in $G$ consists of a path of length at least two together with an edge joining the first and last vertices of the path. A cycle is *Hamiltonian* if it spans $G$. A *cyclic component* of $G$ is a component which is a cycle.

A set of edges in $G$ is a *matching* if no two of them share an endvertex. A *perfect matching* (or 1-*factor*) in $G$ is a matching the edges of which span $G$. A (proper) *edge coloring* of graph $G$ is an assignment of colors to its edges such that all edges of the same color are a matching. The cardinality of any smallest set of colors such that $G$ has a proper edge coloring is called the *edge-chromatic number* (or *chromatic index*) of $G$ and is denoted by $\chi'(G)$. By a classical result of Vizing, every graph $G$ can be edge colored in at most $\Delta(G) + 1$ colors. Hence the set of all graphs is naturally partitioned into two classes, *Class* 1 or *Class* 2, according as to whether $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$, respectively.

The (orientable) *genus* of a graph $G$, $\gamma(G)$, is the smallest genus of any orientable surface in which $G$ can be embedded without edge crossings. The *line graph* of $G$, $L(G)$, is the graph the vertex set of which is the edge set of $G$ and two vertices of $L(G)$ are adjacent if, as edges in $G$, they share a common endvertex. The *complement* of graph $G$, denoted $\overline{G}$, is the graph on the same vertex set as $G$, but in which two vertices are adjacent if and only if they are not adjacent in $G$.

## 2. Degree factors

A factor $F$ of graph $G$ is an *r-factor* if the degree of each vertex in $F$ is $r$. Easily the most studied of degree factors are those in which $r = 1$, i.e., each component is a single edge.

### 2.1. 1-factors

The literature on matchings and 1-factors is vast in its own right and there already exist several sources surveying many of the results in this area. (See, for example, [253,315].) Hence in the present work we shall concentrate mostly on those properties of 1-factors and 1-factorizations which most naturally extend to analogous properties of more general factors.

Historically, one of the first sufficient conditions for a 1-factor was discovered by Petersen [304] and today his result is viewed by most as one of the seminal results in the study of graph factors.

**Theorem 2.1** (*Petersen [304]*). *Every* 2-*edge-connected* 3-*regular multigraph has a* 1-*factor* (*and hence also a* 2-*factor*).

Petersen's result was later generalized by Bäbler as follows:

**Theorem 2.2** (*Bäbler [29]*). *Every* $(r-1)$-*edge-connected r-regular multigraph with an even number of vertices has a* 1-*factor*.

Certainly one of the most influential theorems in the study of 1-factors (at least in general, i.e., non-bipartite, graphs) has been the seminal result called *Tutte's* 1-*factor Theorem*.

**Theorem 2.3** (*Tutte [351]*). *A graph G has a* 1-*factor if and only if for each* $S \subseteq V(G)$, $c_o(G - S) \leqslant |S|$, *where* $c_o(G - S)$ *denotes the number of components of* $G - S$ *which have an odd number of vertices*.

One of the most popular areas of work involving 1-factors has been the search for interesting sufficient conditions for their existence.

A topic which has gained considerable popularity in the last 30 years or so is the study of factor properties in graphs which have certain *forbidden* (induced) subgraphs such as the claw. Sumner [339,340] and Las Vergnas [227] independently proved the next theorem which is of this type.

**Theorem 2.4.** *If G is a connected claw-free graph of even order*, *then G has a* 1-*factor*.

Sumner extended this result as follows.

**Theorem 2.5.** *If G is an n-connected graph of even order and G has no induced subgraph isomorphic to the bipartite graph* $K_{1,n+1}$, *then G has a* 1-*factor*.

A different type of subgraph condition sufficient for the existence of 1-factors was discovered by Fulkerson et al. [119]; see also [256]. Graph $G$ is said to have the *odd cycle property* if every pair of odd cycles in $G$ either have a vertex in common or are joined by an edge.

**Theorem 2.6.** *If G is r-regular of even order and has the odd cycle property*, *then G has a* 1-*factor*.

Yet another sufficient condition, this time topological, is given in the next result due to Nishizeki. Let $\gamma(G)$ denote the (orientable) genus of the graph $G$.

**Theorem 2.7** (*Nishizeki [296,297]*). *If G is a k-connected graph* ($k \geqslant 4$) *of even order and if* $\gamma(G) < k(k-2)/4$, *then G has a* 1-*factor*.

Two other graph parameters which have been considered in connection with 1-factors are toughness and binding number. The *toughness* of graph $G$, denoted by tough $(G)$, is defined to be $+\infty$ when $G$ is complete and otherwise to be

$$\min\{|S|/c(G-S) | S \subseteq V(G)\},$$

where the minimum is taken over all cutsets $S \subseteq V(G)$ and $c(G-S)$ denotes the number of components of $G-S$.

The next result follows immediately from Tutte's 1-factor Theorem.

**Corollary 2.8.** *If G is of even order and tough* $(G) \geqslant 1$, *then G has a* 1-*factor*.

The binding number of G, denoted *bind* $(G)$, is defined to be

$$\min\{|N(X)|/|X| \,|\, \emptyset \neq X \subseteq V(G) \text{ and } N(X) \neq V(G)\}.$$

The next theorem is due to Anderson [22] and can be regarded as a binding number result.

**Theorem 2.9.** *Let G be a graph of even order. If, for all* $X \subseteq V(G)$,

$$|N(X)| \geqslant \min\left\{|V(G)|, \tfrac{4}{3}|X| - \tfrac{2}{3}\right\},$$

*then G has a* 1-*factor*.

The next result is due independently to Kundu [225] and Lovász [251].

**Theorem 2.10.** *There exists a graph G having a* 1-*factor and degree sequence* $d_1, d_2, \ldots, d_n$ *if and only if both the sequences* $d_1, \ldots, d_n$ *and* $d_1 - 1, \ldots, d_n - 1$ *are graphical*.

Highly symmetric graphs of even order are guaranteed to have 1-factors by the next result. (See [253, Theorem 5.5.24] for an extension of this result.)

**Theorem 2.11** (*Little et al. [240]*). *If G is a connected graph of even order the automorphism group of which acts transitively on* $V(G)$, *then G has a* 1-*factor containing any given edge*.

There are many recent papers investigating the existence of 1-factors containing or excluding specified edge sets. However, space does not permit us to treat these results and for the case of 1-factors, we direct the interested reader to three survey articles on the subject [241,311,312]. For some sample sufficient conditions for the existence of other types of factors containing a given edge or edges, see [56,83,182,320,264].

If graph $G$ has exactly one 1-factor, then the following three facts are known about the structure of $G$. The first is due to Kotzig [221], the second to Lovász [253] and the third to Hetyei (unpublished).

**Theorem 2.12.** *Let G be connected and have a unique* 1-*factor. Then*:

(a) *G has a cutedge belonging to the* 1-*factor*;
(b) *G contains a vertex of degree* $\leqslant \lfloor \log_2(|V(G)|+1) \rfloor$; *and*
(c) $|E(G)| \leqslant (|V(G)|/2)^2$.

Gabow, et al. [120,121] invented an $O(|E| \log^4 |V|)$ algorithm to decide whether a graph has a *unique* 1-factor and to find it, if it exists.

Another problem concerning 1-factors which has attracted considerable interest is that of determining how many there are. Let us denote by $\Phi(G)$ the number of 1-factors in graph $G$. It has been shown that one can bound $\Phi(G)$

below by a certain matrix function called a *Pfaffian*. In the case when $G$ is *planar*, the Pfaffian can be used to exactly compute $\Phi(G)$ in polynomial time. (For details, see [253, Section 8.3].)

The connectivity of the graph $G$ can also be employed to yield a lower bound on $\Phi(G)$ in some cases. A graph $G$ is said to be *bicritical* if $G - x - y$ has a 1-factor for every choice of two different vertices $x$ and $y$. (For further reading on bicritical graphs, see [253].)

**Theorem 2.13.** *If $G$ is $k$-connected and has a* 1-*factor*, *then either*

(a) *$G$ has at least $k!$* 1-*factors*, *or else*
(b) *$G$ is bicritical.*

It seems somewhat counterintuitive, perhaps, that bicritical graphs should be the exception here, but it has proven much more difficult to bound $\Phi(G)$ in the bicritical case. Study of the perfect matching polytope of $G$, $PM(G)$, (see [253]) can be utilized to give the bound in the next result.

**Theorem 2.14.** *If $G$ is bicritical*, *then* $\Phi(G) \geqslant |V(G)|/2 + 1$.

From the two preceding facts one has the following.

**Theorem 2.15.** *If $G$ is $k$-connected and contains a* 1-*factor*, *then there exists a function of $k$*, $p_0(k)$, *such that if* $|V(G)| \geqslant p_0(k)$, *then $G$ has at least $k!$* 1-*factors*.

In the special case of bipartite graphs, the history of 1-factors has two principal roots. The first is a result due to Hall [135] and the second, a result due to König [218,219]. (In fact, it can be shown that these two results are equivalent.) First we state Hall's Theorem.

**Theorem 2.16.** *Let $G$ be a bipartite graph with vertex bipartition $V(G) = A \cup B$. Then $G$ has a matching of $A$ into $B$ if and only if* $|N(X)| \geqslant |X|$, *for all* $X \subseteq A$.

An immediate consequence is the following even earlier result due to Frobenius [116] which is usually called the *Marriage Theorem*.

**Theorem 2.17.** *Let $G$ be a bipartite graph with vertex bipartition $V(G) = A \cup B$. Then $G$ has a* 1-*factor matching $A$ onto $B$ if and only if*

(a) $|A| = |B|$ *and*
(b) $|N(X)| \geqslant |X|$, *for all* $X \subseteq A$.

A subset $C \subseteq V(G)$ is called a *vertex cover* of $G$ if every edge of $G$ has at least one endvertex in $C$. The size of any smallest vertex cover in $G$ is denoted by $\tau(G)$ and called the *vertex covering number* of $G$. The size of any largest matching in $G$ is denoted by $\nu(G)$ and called the *matching number* of $G$. It is clear that in any graph $G$, $\nu(G) \leqslant \tau(G)$. However, if $G$ is bipartite, König [218,219] proved the following.

**Theorem 2.18.** *If $G$ is bipartite*, *then* $\nu(G) = \tau(G)$.

Since the above result asserts the equality of the maximum of one quantity and the minimum of another, it is often referred to as a *minimax* theorem. Indeed, in more recent times, especially with the advent of linear programming, such so-called minimax results have gained increasing importance. The study of minimax theorems is outside the scope of this survey, but for an introduction to such ideas at least within the confines of graph theory, and for associated *polytopal* ideas, the reader is referred to [253, Chapters 7 and 12].

For an arbitrary simple bipartite graph, the following lower bound for $\Phi(G)$ is known.

**Theorem 2.19** (*Hall [134]*). *Let $G$ be a simple bipartite graph with bipartition $V(G) = A \cup B$ and assume that for each vertex in $A$ has degree at least $k$. Then if $G$ has at least one* 1-*factor*, *it has at least $k!$* 1-*factors*.

If the bipartite graph $G$ is regular, then we can obtain better bounds on the number of 1-factors. In the case of lower bounds, an important idea from algebra is extremely useful. Let $A=(a_{ij})$ be an $n\times n$ matrix.

The *permanent* of matrix $A$, denoted per$A$, is given by

$$\operatorname{per} A=\sum a_{1\pi(1)}a_{2\pi(2)}\cdots a_{n\pi(n)},$$

where the sum extends over all permutations $\pi$ of the set $\{1,\dots,n\}$.

Note that the only difference between the permanent and the more familiar notion of the determinant of a matrix is that in the case of the permanent, each term is taken with a plus sign. There is a direct correspondence between binary $n\times n$ matrices and simple bipartite graphs with bipartition $V(G)=A\cup B$ where $|A|=|B|$ obtained as follows. Let $A=\{a_1,\dots,a_n\}$ and $B=\{b_1,\dots,b_n\}$ and suppose we identify the rows of an $n\times n$ binary matrix $M$ with the elements of $A$ and the columns of $M$ with the elements of $B$. Moreover, let us define the $(i,j)$ entry of $M$ to be 1 if and only if vertex $a_i$ is adjacent to vertex $b_j$ in $G$, and 0, otherwise. Then it is clear that per $M=\Phi(G)$. Thus, at least in theory, this gives an algorithm for computing *exactly* the number of 1-factors in a simple bipartite graph. However, in fact, the problem of computing the permanent of a matrix is known to be $\#P$-complete! (See [358,359].) (The reader is referred to [124] for a discussion of the concept of $\#P$-completeness and the complexity of enumeration problems.) Thus, the permanent approach is highly unlikely to ever yield a polynomial procedure for exact counting of 1-factors in simple bipartite graphs. On the other hand, however, equivalence between the permanent and the number of 1-factors led to the discovery of a non-trivial lower bound for $\Phi(G)$ when $G$ is bipartite and regular.

**Theorem 2.20.** *Let $G$ be a simple $k$-regular bipartite graph on $2n$ vertices. Then*

$$n\left(\frac{k}{n}\right)^n\leqslant\Phi(G)\leqslant(k!)^{n/k}.$$

The first inequality in the above theorem is equivalent to the famous *van der Waerden Conjecture* on permanents which was formulated in 1926 [360] and proved independently by Falikman [105] and Egoryčev [89,90]. The second inequality was proved by Brégman [48]. The upper bound is known to be sharp whenever $k|n$. There has been considerable interest in improving the lower bound especially for fixed $k$ and large $n$. Schrijver and Valiant [330,327] conjectured the new lower bound below and in 1998 Schrijver verified the conjecture.

**Theorem 2.21** (*Schrijver [329]*). *If $G$ is a $k$-regular bipartite graph of order $2n$, then*

$$\Phi(G)\geqslant\left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n.$$

### 2.2. $(g,f)$-factors

Let $G$ be a finite multigraph with loops and let $f,g$ be mappings of $V(G)$ into the non-negative integers.

A $(g,f)$-*factor* of $G$ is a spanning subgraph $F$ such that $g(v)\leqslant\deg_F(v)\leqslant f(v)$ for all $v\in V(G)$.

Note that If $f\equiv g\equiv 1$, then a $(g,f)$-factor (i.e., $(1,1)$-factor) is just a 1-factor.

The next result, due to Lovász, is known as the $(g,f)$-*factor theorem*. Let $e_G(A,B)$ denote the number of edges in graph $G$ joining vertex sets $A$ and $B$.

**Theorem 2.22** (*Lovász [250]*). *Graph $G$ has a $(g,f)$-factor if and only if*

$$f(D)-g(S)+\deg_{G-D}(S)-\hat{q}_G(D,S,g,f)\geqslant 0$$

*for all pairs of disjoint sets $D,S\subseteq V(G)$, where $\hat{q}_G(D,S,g,f)$ denotes the number of components $C$ of $G-(D\cup S)$ having $g(v)=f(v)$ for all $v\in V(C)$ and $e_G(V(C),S)+f(V(C))\equiv 1 \pmod 2$.*

Due to the complicated nature of the $(g,f)$-factor theorem, it has proven difficult to use in its full generality, but rather the bulk of results in this direction treat special cases only. For example, the next sample result shows that if one assumes that $g(v)\leqslant 1$, for all $v\in V(G)$, the existence criterion for a $(g,f)$-factor simplifies somewhat.

**Theorem 2.23** (*Las Vergnas [228]*). *Let G be a graph and f and g two integer-valued functions defined on $V(G)$ such that $0 \leqslant g(x) \leqslant 1 \leqslant f(x)$. Then G contains a $(g, f)$-factor if and only if for every subset $X \subseteq V(G)$, $f(X)$ is at least equal to the number of components C of $G[V - X]$ such that either $C = \{x\}$ and $g(x) = 1$, or $|C|$ is odd and $\geqslant 3$ and $g(x) = f(x) = 1$ for all $x \in C$.*

Several related sufficiency-type theorems can be found in [84]. We state one of them.

**Theorem 2.24.** *Let G be a graph and f and g functions from $V(G)$ to the non-negative integers such that $g(v) \leqslant \deg_G(v)$, $0 \leqslant f(v)$ and $g(v) < f(v)$, for all $v \in V(G)$. If*

$$\frac{g(x)}{\deg_G(x)} \leqslant \frac{f(y)}{\deg_G(y)},$$

*for every pair of adjacent vertices x and y in G, then G has a $(g, f)$-factor.*

See also [27] for more sufficient conditions for the existence of a $(g, f)$-factor, [25,141] for simplified existence theorems for such factors and [183,242,188,243] for the existence of such a factor having additional properties such as including or excluding prescribed sets of edges.

Once again, let f and g be two functions from $V(G)$ into the positive integers such that $g(v) \leqslant f(v)$ for all $v \in V(G)$ and suppose that there exists another function h from $V(G)$ to the positive integers such that $g(v) \leqslant h(v) \leqslant f(v)$ for every vertex $v \in V(G)$ and $h(V(G))$ is even. Graph G is said to have *all $(g, f)$-factors* if and only if G has an h-factor for every h such that $g(v) \leqslant h(v) \leqslant f(v)$ for all $v \in V(G)$.

**Theorem 2.25** (*Niessen [287]*). *Let G be a multigraph and let g and f be as above. Then G has all $(g, f)$-factors if and only of*

$$g(D) - f(S) + d_{G-D}(S) - q^*_G(D, S, g, f) \geqslant \begin{cases} -1 & \text{if } f \neq g, \\ 0 & \text{if } f = g, \end{cases}$$

*for all disjoint sets $D, S \subset V(G)$, where $q^*_G(D, S, g, f)$ denotes the number of components C of $G - (D \cup S)$ such that there exists a vertex $v \in V(C)$ with $g(v) < f(v)$ or $e_G(V(C), S) + f(V(C)) \equiv 1 \pmod 2$.*

It is apparently unknown whether there is a polynomial algorithm to test if a graph G has all $(g, f)$-factors.

An exciting alternative approach to $(g, f)$-factors is provided by the concept of a "fractional" $(g, f)$-factor. Let G be a graph without loops in which each edge e has multiplicity $c_e$. As usual, suppose two mappings g and f of $V(G)$ into the non-negative integers are given with $g(v) \leqslant f(v)$, for all vertices $v \in V(G)$. A vector $x = (x_e)$ with $|E(G)|$ real components such that $0 \leqslant x_e \leqslant c_e$ and $g(v) \leqslant \deg_x(v) \leqslant f(v)$ for all v is called a *fractional* $(g, f)$-factor. Here, $\deg_x(v)$ is defined to be the sum of the values of all $x_{uv}$ where uv is an edge incident with v.

An important feature of fractional factors is that they can be studied using network flow theory and its accompanying polynomial algorithms. Moreover, in some cases, fractional $(g, f)$-factors can be transformed into integral $(g, f)$-factors.

**Theorem 2.26** (*Anstee [26]*). *Let G, g and f be as above and suppose $G_{g=f}$ denotes the subgraph of G induced by the vertices upon which $g(v) = f(v)$. Suppose also that G, g and f satisfy either of the properties* (i) *or* (ii) *below.*

(i) *$G_{g=f}$ is bipartite, or*
(ii) *$g = f$, $\sum_v f(v) \equiv 0 \pmod 2$ and every pair of vertex-disjoint odd cycles are joined by an edge.*

*Then G has a $(g, f)$-factor if and only if G has a fractional $(g, f)$-factor.*

The above theorem, in turn, can be used to give a very quick proof of the following result on graphs with the odd cycle property. (A *k-factor* is a factor in which all degrees $= k$. Further results about k-factors will be found in Section 2.5.)

**Theorem 2.27** (*Anstee [26]*). *Let G be a graph with the odd cycle property. Suppose G has a k-factor. Then*

(i) *for any even integer $r \leqslant k$, or*
(ii) *when $|V(G)|$ is even, for any odd integer r satisfying $1 \leqslant r < k$,*

*G has an r-factor.*

For further information on the connections between network flows and graph factors, as well as fractional $(g, f)$-factors, see [110–114,215,244,245,372,378,234,255]. Finally, [148] contains an interesting application of $(g, f)$-factors in graphs to statistical data analysis. Fractional matchings have also been studied in [280,46,314,246]. See also [315] and [253].

### 2.3. $[a, b]$-factors

Let $a$ and $b$ be integers such that $1 \leqslant a \leqslant b$. An $[a, b]$-*factor* $H$ of graph $G$ is a factor of $G$ for which $a \leqslant \deg_H(v) \leqslant b$, for all $v \in V(G)$.

Of course, $[a, b]$-factors are just a special case of $(g, f)$-factors, but an important one nonetheless.

A sufficient degree condition for the existence of an $[a, b]$-factor was derived by Li and Mao-cheng [238] and is presented in the next theorem. This result generalizes previous results of Iida and Nishimura [170] and Nishimura [293].

**Theorem 2.28** (*Li and Mao-cheng [238]*). *Let G be a graph of order n and let a and b be integers such that $1 \leqslant a < b$. Then if $\delta(G) \geqslant a$, $n \geqslant 2a + b + (a^2 - a)/b$ and*

$$\max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{an}{a+b},$$

*for any two non-adjacent vertices x and y of G, G has an $[a, b]$-factor.*

Kano [187] obtained the following result involving a criterion having binding number flavor.

**Theorem 2.29.** *Let a and b be integers such that $2 \leqslant a < b$ and suppose G is a graph of order at least $6a + b$. Define $\lambda = b/(a + b - 1)$. Suppose that for all $X \subseteq V(G)$, $N(X) = V(G)$ if $|X| \geqslant \lfloor n\lambda \rfloor$, and $\lambda|N(X)| \geqslant |X|$, if $|X| < \lfloor n\lambda \rfloor$, then G has an $[a, b]$-factor.*

There are a number of results giving sufficient conditions for special graph classes to have $[a, b]$-factors.

**Theorem 2.30** (*Kano and Saito [192]*). *Suppose $k$, $r$, $s$ and $t$ are integers such that $0 \leqslant k \leqslant r$ and $1 \leqslant t$. If $ks \leqslant rt$, then an $[r, r + s]$-graph has a $[k, k + t]$-factor.*

The following gives a sufficient condition for an $[a, b]$-factor in the line graph of $G$ in terms of the minimum degree and the independence number of $G$.

**Theorem 2.31** (*Nishimura [294]*). *If G is a and a and b are integers such that $1 \leqslant a < b$, then if $\delta(G) \geqslant (\alpha(G)/2) + 1$, the line graph $L(G)$ has an $[a, b]$-factor.*

Tokuda [344] and Li [232] have derived a minimum degree criterion and a neighborhood union criterion, respectively, sufficient for the existence of an $[a, b]$-factor in $K_{1,n}$-free graphs.

The next result is very similar in form to Tutte's 1-factor Theorem. Let $i(G)$ denote the number of isolated vertices of graph $G$.

**Theorem 2.32** (*Las Vergnas [228], Amahashi and Kano [21], Gunther et al. [129]*). *Let $n \geqslant 2$ be an integer. The following three statements are equivalent*:

(i) *Graph G has a $[1, n]$-factor,*

(ii) $i(G-S) \leqslant n|S|$, *for all* $S \subseteq V(G)$ *and*
(iii) $|U| \leqslant n|N(U)|$, *for all independent subsets* $U \subseteq V(G)$.

An important special subclass of $[1, n]$-factors known as *star factors* is discussed in Section 3.

The next two results deal with "almost regular" factors. A $[k, k+1]$-factor is sometimes called an *almost regular* (or *semiregular*) factor.

**Theorem 2.33** (*Thomassen [342]*). *If G is an* $[r, r+1]$-*graph, then G has a* $[k, k+1]$-*factor for all k*, $0 \leqslant k \leqslant r$.

**Theorem 2.34** (*Kano [186]*). *Let* $r \geqslant 3$ *be an odd integer and let k be an integer such that* $1 \leqslant k \leqslant (2r/3) - 1$. *Then every r-regular graph has a* $[k, k+1]$-*factor each component of which is regular.*

*2.4. f-factors*

Let $G$ be a multigraph possibly with loops and $f$, a non-negative, integer-valued function on $V(G)$. Then a spanning subgraph $H$ of $G$ is called an *f-factor* of $G$ if $\deg_H(v) = f(v)$, for all $v \in V(G)$. (In other words, an $f$-factor is just an $(f, f)$-factor.)

The next result is called *Tutte's f-factor theorem*. (See [352] for a proof for locally finite graphs and [353] for a somewhat shorter proof for the case when the graph is finite.) But first we need some additional notation. If $f$ is a non-negative integer valued function on $V(G)$, let $\hat{f}$ be defined by $\hat{f}(v) = \deg_G(v) - f(v)$. If $X, Y \subseteq V(G)$, let $\nabla(X, Y)$ denote the set of edges joining $X$ and $Y$.

**Theorem 2.35.** *Graph G has an f-factor if and only if for every two disjoint subsets X and Y of* $V(G)$, *the number of components K of* $G - X - Y$ *for which* $f(V(K)) + |\nabla(X, Y)|$ *is odd, does not exceed* $f(X) + \hat{f}(Y) - |\nabla(X, Y)|$.

It is an interesting fact that although Lovász [250] generalized the $f$-factor theorem of Tutte via his $(g, f)$-factor theorem, it was shown later by Tutte [356] that, in fact, conversely, the $(g, f)$-factor theorem can be derived from the $f$-factor theorem. Thus, in a sense the two results are equivalent.

Recall that by the 1-factor Theorem of Tutte, we know that if a graph $G$ has no 1-factor, then there must exist a set $S \subseteq V(G)$ such that $c_o(G-S) > |S|$. Such a set $S$ is called a *1-barrier* or *antifactor set*. So Tutte's 1-factor theorem could be restated to say that a graph has a 1-factor if and only if it has no 1-barrier. An alternate form of the $f$-factor theorem, similar in form to this restatement of the 1-factor Theorem, was derived by Tutte [352,353, see also 355], using the concept of an *f-barrier*, where an $f$-barrier is a generalization of the idea of a 1-barrier. We do not treat the details here but instead simply state this result.

**Theorem 2.36** (*Tutte [352,353]*). *A graph G has an f-factor if and only if it does not have an f-barrier.*

Fulkerson et al. [119] also found a necessary and sufficient condition for a graph with the odd cycle property to have an $f$-factor, and used integer programming to prove their result. Mahmoodian [256] later showed that this result was a corollary to Tutte's $f$-factor theorem. See also [226].

It is important to know that the $f$-factor problem can, in a sense, be reduced to the 1-factor problem and this reduction was known to Tutte already in 1954 [353]. More particularly, there exists a procedure for reducing the $f$-factor problem on a graph $G$ to the 1-factor problem on a larger graph $G'$. (We will not go into the details here, but instead refer the interested reader to Chapter 10 of [253] or Chapter 2 of [36].)

In many cases, by virtue of the next theorem, due to Kotani, one can reduce the question of the existence of an $f$-factor in a graph $G$ to the same question applied to a collection of its induced subgraphs of fixed order.

**Theorem 2.37** (*Kotani [220]*). *Let G be a connected graph and let p be an integer such that* $0 < p < |V(G)|$. *Let f be an integer-valued function on* $V(G)$ *such that* $2 \leqslant f(v) \leqslant \deg_G(v)$ *for all* $v \in V(G)$. *If every connected induced subgraph of order p of G has an f-factor, then G has an f-factor, or else* $\sum_v f(v)$ *is odd.*

Sufficient conditions for an $f$-factor in terms of $\delta(G)$, of the binding number of $G$ and of the connectivity and independence number of $G$, respectively, are provided by the next three results.

**Theorem 2.38** (*Katerinis and Tsikopoulos [205]*). *Let G be a graph and $a \leqslant b$, two positive integers. Suppose further that*

(i) $\delta(G) \geqslant \dfrac{b}{a+b}|V(G)|$, *and*
(ii) $|V(G)| > \dfrac{a+b}{a}(b+a-3)$.

*Then if f is a function from $V(G)$ to $\{a, a+1, \ldots, b\}$ such that $\sum_v f(v)$ is even, G has an f-factor.*

**Theorem 2.39** (*Kano and Tokushige [193]*). *Let G be a connected graph of order n, let a and b be two integers such that $1 \leqslant a \leqslant b$ and $2 \leqslant b$, and let $f : V(G) \to \{a, a+1, \ldots, b\}$ be a function such that $\sum_v f(v)$ is even. Then if the binding number of G is greater than $(a+b-1)(n-1)/(an-(a+b)+3)$ and $n \geqslant (a+b)^2/a$, G has an f-factor.*

**Theorem 2.40** (*Katerinis and Tsikopoulos [206]*). *Let G be a graph, a and b two positive integers with $a \leqslant b$ and $2 \leqslant b \leqslant 3$ and suppose that*

(i) $\kappa(G) \geqslant \dfrac{2(b-1)}{a}\alpha(G)$ *and*
(ii) $|V(G)| \geqslant 8$.

*If f is a function from $V(G)$ to the positive integers such that*

(iii) $\sum_{x \in V(G)} f(x)$ *is even and*
(iv) $a \leqslant f(x) \leqslant b$ *for every* $x \in V(G)$,

*then G has an f-factor.*

In the special case when *G* has an *f*-factor such that $f(v)$ is odd, for all $v \in V(G)$, *G* is said to have an *odd f-factor*. Amahashi has found a simple Tutte-like criterion for a graph to have an odd *f*-factor.

**Theorem 2.41** (*Amahashi [20]*). *Let G be a graph and n a positive integer. Then G has a $\{1, 3, \ldots, 2n-1\}$-factor if and only if*

$$c_o(G-S) \leqslant (2n-1)|S|,$$

*for every $S \subseteq V(G)$.*

See also [71,349,190]. In [191] one finds a polynomial algorithm to find an odd *f*-factor which is in a sense optimal in terms of its deficiency at each vertex.

## 2.5. *k-factors*

An *f*-factor for which $f(v) = k$, for all $v \in V(G)$, is called a *k-factor*. Petersen, by virtue of a second theorem in his 1891 paper, was present at the creation of the study of *k*-factors as well.

**Theorem 2.42** (*Petersen [304]*). *A graph G has a factorization into 2-factors if and only if it is regular of even degree.*

The analogous result for regular graphs of *odd* degree did not appear until almost 50 years later and is due to Bäbler.

**Theorem 2.43** (*Bäbler [29]*). *Every 2-edge-connected $(2k+1)$-regular multigraph contains a 2-factor.*

A thorough survey tracing the descendants of Petersen's factorization results for regular graphs may be found in Volkmann [362].

There is now an extensive literature on the subject of sufficient conditions for the existence of a $k$-factor. We present only a sampling.

Chvátal [61], who invented the graph parameter called toughness, conjectured the following result which was proved in [97].

**Theorem 2.44.** *If tough $(G) \geqslant k$, then $G$ has an $k$-factor.*

(Katerinis [203] later generalized this result to one about $f$-factors.)

A sufficient condition for an $k$-factor, in terms of $\kappa(G)$ and $\alpha(G)$, is found in the next result. This condition is reminiscent of the well-known Chvátal–Erdős condition for the existence of a Hamilton cycle.

**Theorem 2.45** (*Nishimura [290,291]*). *Let $G$ be a graph and $k$, an even non-negative integer. If*

$$\kappa(G) \geqslant \max\{k(k+2)/2, (k+2)\alpha(G)/4\},$$

*then $G$ has an $k$-factor.*

(See also [201,286].)

The condition involving degree sums given in the next theorem is often called an *Ore condition* after Oystein Ore who first introduced a condition of this type and showed it to be sufficient for the existence of a Hamilton cycle. In [207,79,345,365,229], one finds theorems with hypotheses involving the binding number or conditions suggesting the binding number. We shall be content to state the following result which involves both an hypothesis suggesting the binding number and a second hypothesis which is an Ore condition.

**Theorem 2.46** (*Lenkewitz and Volkmann [229]*). *Let $k \geqslant 2$ be an integer and $G$ a graph of order $n$ with $n \geqslant 4k-6$ and $\delta(G) \geqslant k$. If $k$ is odd, then $n$ is even and $G$ is connected. Let $G$ satisfy*

$$|N(X)| \geqslant \frac{1}{2k-1}(|X| + (k-1)n - 1),$$

*for every independent subset $X$ of $V(G)$ with $|X| \geqslant 2k$, and*

$$\deg(u) + \deg(v) \geqslant \frac{2k-2}{2k-1}n + \frac{4k-5}{2k-1}.$$

*Then $G$ has a $k$-factor.*

Again, following analogous results for the existence of Hamilton cycles, the next result gives a sufficient condition for the existence of a $k$-factor in terms of what is called a *neighborhood union condition*.

**Theorem 2.47** (*Iida and Nishimura [171]*). *Let $k \geqslant 2$ be an integer and let $G$ be a connected graph of order $n$, minimum degree at least $k$ and suppose $kn$ is even. Suppose further that $n \geqslant 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$. Then if $|N_G(u) \cup N_G(v)| \geqslant (1/2)(n+k-2)$ for each pair of non-adjacent vertices $u$ and $v$, $G$ has a $k$-factor.*

Matsuda [263] later extended this result by giving a neighborhood union condition sufficient for the existence of an $[a, b]$-factor and his result was, in turn, further generalized by Li. (See [233].)

Yet another kind of bound involving the degrees of non-adjacent vertices is given next.

**Theorem 2.48** (*Nishimura [293]*). *Let $G$ be a connected graph of order $n$ and let $k$ be an integer $\geqslant 3$ such that $kn$ is even, $n \geqslant 4k-3$ and $\delta(G) \geqslant k$. Then if $\max\{d(u), d(v)\} \geqslant n/2$, for all pairs of non-adjacent vertices $u$ and $v$, $G$ has a $k$-factor.*

We next present a minimum degree condition sufficient to guarantee a $k$-factor in a claw-free graph. (See also [107].)

**Theorem 2.49** (*Egawa and Ota [86]*). *If G is a connected claw-free graph with* $\delta(G) \geqslant \lceil (9k+12)/8 \rceil$ *and if* $k|V(G)|$ *is even*, *then G has a k-factor*.

The following result gives a condition on the minimum degree in claw-free graphs sufficient to yield a 2-factor with a bounded number of cyclic components.

**Theorem 2.50** (*Faudree et al. [106]*). *If G is claw-free with* $\delta(G) \geqslant 4$, *then G has a* 2*-factor with at most* $[6n/(\delta(G)+2)] - 1$ *components. Moreover*, *there is an* $O(n^3)$ *algorithm to construct such a* 2*-factor*.

(See also [323,47] and [57].)

A well-known conjecture about 2-factors is known as *El-Zahár's Conjecture.*

**Conjecture 2.51** (*El-Zahár [95]*). If G is a graph with $n = n_1 + \cdots + n_k$ vertices and $\delta(G) \geqslant \lceil n_1/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then $G$ has a 2-factor in which the cycles have lengths $n_1, \ldots, n_k$, respectively.

El-Zahár himself proved the conjecture true in the case $k = 2$ and Corrádi and Hajnal [69] proved it true when each $n_i = 3$. Further partial results can be found in [179]. We understand that Abbasi [1] has settled the conjecture for $n = |V(G)|$ sufficiently large, but to the best of our knowledge, no proof has yet been published.

Kundu was able to generalize his 1-factor theorem about degree sequences to the $k$-factor case.

**Theorem 2.52** (*Kundu [225]*). *If k is a positive integer and the sequences* $d_1, \ldots, d_n$ *and* $d_1 - k, \ldots, d_n - k$ *are both graphical*, *then* $d_1, \ldots, d_n$ *can be realized by a graph G which contains a k-factor*.

Kleitman and Wang [214] give an alternate proof of Kundu's result and also give a polynomial algorithm for constructing such a graph $G$ containing a $k$-factor.

The next three results yield conditions on vertex-deleted subgraphs sufficient to guarantee the existence of a $k$-factor in the parent graph.

**Theorem 2.53** (*Egawa et al. [81]*). *Let G be a connected graph and p be an integer such that* $0 < p < |V(G)|$. *Suppose* $k|V(G)|$ *is even and* $G - V(P)$ *has a k-factor for each connected induced subgraph P of order p. Then G has a k-factor*.

**Theorem 2.54** (*Saito [324]*). *Suppose G is a graph with a* 1*-factor F and order at least four and let k be a positive integer. Then if* $G - \{u, v\}$ *has a k-factor for each edge* $uv \in F$, *G itself has a k-factor*.

(See [101] for a generalization of the above result.)

A graph $G$ is *hypohamiltonian* (respectively, *hypotraceable*) if $G$ does not have a Hamilton cycle (respectively, path), but $G - v$ does, for all $v \in V(G)$.

**Theorem 2.55** (*Katerinis [199]*). *If G is either hypohamiltonian or hypotraceable*, *then G has a* 2*-factor*.

A sufficient condition for a $k$-factor in the line graph is given next.

**Theorem 2.56** (*Nishimura [292]*). *If* $k \geqslant 2$ *is an integer and G is a connected graph with* $k|E(G)|$ *even and if* $\delta(L(G)) \geqslant (9k+12)/8$, *then* $L(G)$ *has a k-factor*.

A sample result on the existence of $k$-factors in bipartite graphs is the following.

**Theorem 2.57** (*Katerinis [204]*). *Let G be a bipartite graph with bipartition* $V(G) = X \cup Y$ *and k be a positive integer. Then if*:

(i) $|X| = |Y|$,
(ii) $\delta(G) \geqslant \lceil |X|/2 \rceil \geqslant k$, *and*
(iii) $|X| \geqslant 4k - 4\sqrt{k} + 1$, *when* $|X|$ *is odd and* $|X| \geqslant 4k - 2$, *when* $|X|$ *is even*, *then G has a k-factor*.

Hall-type conditions sufficient for a bipartite graph to have a 2-factor (respectively, $k$-factor) may be found in [202] (respectively, [100]). Conditions for the existence of $k$-factors in multipartite graphs have been obtained by Hoffmann [160,161] and by Hoffmann and Rodger [159].

Suppose $G$ is $r$-regular and has edge-connectivity $\lambda$. If $G$ is a multigraph, all values of $k$ for which $G$ is guaranteed to have a $k$-factor are known [42]. Similarly, all such $k$ are known when $G$ is simple [288].

Katerinis [200] has obtained the following "interpolation" theorem about $k$ factors.

**Theorem 2.58.** *Let $m, \ell, n$ be three odd integers such that $m < \ell < n$. Then if graph $G$ has an $m$-factor and an $n$-factor, it also has an $\ell$-factor.*

A recent interesting generalization of the above result may be found in [28].

Bermond and Las Vergnas [32] showed that in a graph which is not regular, but is "sufficiently close" to being regular, one can guarantee the existence of odd and even $k$-factors.

In a very recent paper, Hoffmann and Volkmann [165] prove the following result about $k$-factors in graphs with small diameter.

**Theorem 2.59.** *A connected $d$-regular graph with $d \geqslant 2$ and diameter $\leqslant 3$, has every $k$-factor for $k|V(G)|$ even.*

Hendry [149] initiated the study of graphs with *unique* $k$-factors and his conjecture on the maximum number of edges that such a graph may have was proved by Johann [177,178]. Results on the structure of a bipartite graph possessing a unique $k$-factor may be found in [163,164].

Jackson and Whitty proved an interesting result about degrees in the presence of a unique $f$-factor.

**Theorem 2.60** (*Jackson and Whitty [173]*). *If $G$ is a 2-edge-connected graph with a unique $f$-factor $F$, then some vertex has the same degree in $F$ as in $G$.*

In [120,121], the authors show that the $O(|E|\log^4|V|)$ algorithm for unique 1-factors can be extended to finding a unique $f$-factor, if one exists. The running time of the extended algorithm remains the same as that for the 1-factor case.

### *2.6. Connected factors*

It is also of interest to investigate when one can be assured of the existence of a *connected* factor. For example, a Hamilton cycle is just a connected 2-factor.

**Theorem 2.61** (*Kano [189]*). *Let $k$ be a positive integer and let $G$ be a connected graph of order $n$ and minimum degree greater than $k$ where $kn$ is even and $n \geqslant 4k-5$. If for each pair of non-adjacent vertices $u$ and $v$ of $G$, $\deg(u)+\deg(v) \geqslant n$, then $G$ has both a Hamilton cycle $C$ and a $k$-factor $F$. Hence $G$ has a* connected $[k, k+2]$*-factor.*

Matsuda [266,267] has extended the above result to the cases of $[a, b]$-factors (and $k$-factors) containing a given Hamilton cycle.

Kano conjectured that the hypotheses of the above theorem were sufficient to guarantee the existence of a $[k, k+1]$-factor and this follows from the next result.

**Theorem 2.62** (*Li and Mao-cheng [237]*). *Let $G$ be a connected graph of order $n$ and let $f$ and $g$ be two functions from $V(G)$ to the positive integers which satisfy $2 \leqslant g(v) \leqslant f(v)$ for each vertex $v \in V(G)$. Suppose $G$ has an $(g, f)$-factor $F$ and put $\mu = \min\{g(v)|v \in V(G)\}$. Suppose that among any three independent vertices of $G$ there are (at least) two vertices with degree sum at least $n - \mu$. Then $G$ has a matching $M$ such that $M$ and $F$ are edge-disjoint and $M \cup F$ is a connected $(g, f+1)$-factor.*

(See also [258,259].)

**Theorem 2.63** (*Li and Liu [230]*). *If $G$ is a 2-connected claw-free graph, then $G$ has a connected $[2, 3]$-factor.*

**Theorem 2.64** (*Li et al.* [231]). *If $G$ is a 2-connected claw-free graph containing a $k$-factor where $k \geqslant 2$, then $G$ contains a connected $[k, k+1]$-factor.*

**Theorem 2.65** (*Xu et al.* [368]). *Let $n \geqslant 3$ be an integer and let $G$ be a $K_{1,n}$-free graph. Suppose $f$ and $g$ are positive integer-valued functions defined on $V(G)$ such that $g(v) \leqslant f(v)$, for all $v \in V(G)$. Then if $G$ has a $(g, f)$-factor, $G$ has a* connected *$(g, f+n-1)$-factor.*

**Theorem 2.66** (*Kouider and Mahéo* [223]). *Let $G$ be a connected graph of order $n$ and minimum degree $\delta$. Let $a$ and $b$ be two integers such that $2a \leqslant b$. Suppose further that $n \geqslant ((a+b)(a+b-1))/b$ and $\delta(G) \geqslant n/(1+\lfloor b/a \rfloor)$, then $G$ has a connected $[a, b]$-factor.*

(See also [93] and [281].)

In [235] and [236] the authors derive a degree sum condition for graphs containing a Hamiltonian cycle $H$ which is sufficient to guarantee the existence of a $[k, k+1]$-factor containing $H$. (See also [265].)

Tutte [354] discovered a simple criterion for a graph to decompose into a prescribed number of *connected* factors.

**Theorem 2.67.** *A graph $G$ decomposes into $n$ connected factors if and only if for all $\alpha \geqslant 0$ deleting any set of $\alpha$ edges from $G$ leaves a graph with at most $1 + \alpha/n$ components.*

We will not go further in the direction of connected factors, but refer the reader to a survey on the subject of connected graph factors (including spanning trees) by Kouider and Vestergaard [224].

## 3. Component factors

In a sense, the results of Section 2.6 above provide a nice transition to this section. There the issue was the existence of a type of degree factor which had only one component, namely a degree factor which was connected. In this section we address problems concerning factors each component of which is described by properties other a degree bound. The next definition will serve as a good beginning point.

Let $\{G_1, G_2, \ldots, G_k\}$ be an arbitrary set of $k$ graphs. Then graph $G$ is said to have a *$\{G_1, G_2, \ldots, G_k\}$-factor* if $G$ has a factor each component of which isomorphic to some member of $\{G_1, G_2, \ldots, G_k\}$. (Here we wish to make it clear that the components of the factor sought may be isomorphic to *different* members of $\{G_1, G_2, \ldots, G_k\}$.)

For example, suppose it is required that each component of the factor be a path. Such a factor is called a *path factor*. Recall that $P_i$ denotes a path having exactly $i$ vertices. It is of interest to note that by Theorem 2.32, a graph $G$ has a path factor if and only if $i(G-S) \leqslant 2|S|$, since the following three statements are clearly equivalent:

(i) $G$ has a path factor,
(ii) $G$ has a $\{P_2, P_3\}$-factor and
(iii) $G$ has a $\{K_{1,1}, K_{1,2}\}$-factor.

Kaneko [180] proved part (a) of the following. Part (b) was obtained in [209].

**Theorem 3.1.** (a) *Every 3-regular graph has a $\{P_3, P_4, P_5\}$-factor.*
(b) *Every 2-connected 3-regular graph has a $\{P_3, P_4\}$-factor.*

An interesting conjecture along this line, due to Akiyama and Kano [11,12], remains open.

**Conjecture 3.2.** Every 3-connected 3-regular graph of order divisible by three has a $P_3$-factor.

The next result gives a sufficient condition for a graph to have a $P_3$-factor in the case in which it is claw-free.

**Theorem 3.3** (*Kaneko et al.* [181]). *Let $G$ be a claw-free graph with $|V(G)| \equiv 0 \bmod 3$ and having at most two endblocks. Then $G$ has a $P_3$-factor.*

A bit more generally, one can ask when there exists a path factor each component of which has a prescribed minimum length. For example, Kaneko [180] has found a necessary and sufficient condition for a graph to have a factor each component of which is a path on at least three vertices. This result, which involves a Tutte-like condition, is perhaps the first characterization of graphs which have a path factor not including $K_2 = P_2$.

A graph $G$ is said to be *factor-critical* if $G - v$ has a perfect matching for every vertex $v \in V(G)$. Let $G$ be a factor-critical graph with at least three vertices and suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. Add $n$ new vertices $\{w_1, w_2, \ldots, w_n\}$ to $G$ together with edges $v_i w_i$, for $1 \leqslant i \leqslant n$. The resulting graph $H$ on $2n$ vertices is called a *sun*. Let $c_s(G)$ denote the number of components of $G$ which are sun graphs. Then Kaneko's result can be stated as follows.

**Theorem 3.4.** *A graph $G$ has a path factor in which every component path has length at least two if and only if $c_s(G - S) \leqslant 2|S|$, for every subset $S \subseteq V(G)$.*

For claw-free graphs, one has the next result.

**Theorem 3.5** (*Ando et al. [23]*). *Let $d$ be a non-negative integer and let $G$ be a claw-free graph with $\delta(G) \geqslant d$. Then $G$ has a path factor in which all paths have at least $d + 1$ vertices.*

An $F_c$*-factor* is a spanning subgraph in which each component is a single edge or an odd cycle. (Note that the terminology "$F_c$-factor" is our own. These factors were originally called "$F$-factors", but the terms "$F$-factor" and "$F$-factorization problem" have subsequently come to mean something else and hence we will reserve them for a different use later on in this paper.) The next result involving $F_c$-factors can be viewed as a generalization of Hall's Theorem on perfect matchings to the non-bipartite case.

**Theorem 3.6** (*Steinparz [338]*). *A graph $G$ has an $F_c$-factor if and only if $|N(S)| \geqslant |S|$, for every $S \subseteq V(G)$.*

Another type of component factor which has been studied is the "star". A *star* is a subgraph isomorphic to the complete bipartite graph $K_{1,n}$, for any $n \geqslant 1$. Let $\mathscr{S}$ denote the family of all stars having at least one edge and let $\mathscr{S}_n$ denote the family of all stars having at least one and at most $n$ edges. Las Vergnas, Amahashi and Kano, Egawa, Kano and Kelmans, Saito and Watanabe, Gunther, Hartnell and Rall, Chen, Egawa and Kano, as well as Hell and Kirkpatrick, among others, have studied so-called *star* factors, that is, factors each component of which is a star.

The first statement in the following theorem involves a criterion similar in form to Tutte's 1-factor criterion. Here $i(G)$ denotes the number of isolated vertices of graph $G$.

**Theorem 3.7** (*Las Vergnas [228], Amahashi and Kano [21], Gunther et al. [129]; see also Hell and Kirkpatrick [145]*). *Let $n$ be an integer greater than or equal to 2. Then the following statements are equivalent*:

(1) *$G$ has an $\mathscr{S}_n$-factor*,
(2) *$i(G - S) \leqslant n|S|$ for every $S \subseteq V(G)$*,
(3) *$|N(U)| \geqslant (1/n)|U|$ for every independent set $U \subseteq V(G)$*.

(See also Section 2.3 above.)

Yu [373] has studied the "barriers" involved, i.e., the sets $S$ which fail to satisfy condition (2) above. He has also shown [374] that every regular graph has the property that every edge lies in some $\mathscr{S}$-factor. See also [55] for a related result for not necessarily regular graphs and [54] for an analogous result for the situation when the degrees of the stars are bounded above by a given function on the vertices.

Let us now call a star factor of a graph $G$ *strong* if each of the stars is an induced subgraph of $G$.

**Theorem 3.8** (*Kelmans [210], Saito and Watanabe [325]*). *A connected graph $G$ has a strong $\mathscr{S}$-factor if and only if $G$ is not an odd-cactus*.

(A graph is a *cactus* (or *clique tree*) if it is connected and each of its blocks is complete and a cactus (clique tree) is *odd* if each of its blocks has odd order.)

**Theorem 3.9** (*Egawa et al.* [85]). *Let $n \geqslant 2$ be an integer. Then a graph $G$ has a strong $\mathscr{S}_n$-factor if and only if $c_{oc}(G-S) \leqslant n|S|$ for all subsets $S \subseteq V(G)$, where $c_{oc}(G-S)$ denotes the number of odd cactus components of $G-S$.*

Yu has studied graphs with a unique $\mathscr{S}_n$-factor and in addition, has proved the following result.

**Theorem 3.10** (*Yu* [375]). *Let $r \geqslant 4$ be an integer and let $G$ be a connected $r$-regular graph of order $n$ which is not isomorphic to $K_{r,r}$. Then $G$ has at least $n$ star factors each of which is either a proper $\mathscr{S}_r$-factor or a proper $\mathscr{S}_{r-1}$-factor.*

(Here a $\mathscr{S}_r$-factor is *proper* if it has at least one component isomorphic to $K_{1,r}$).

In the special case of $\mathscr{S}_4$-factors in which every component is isomorphic to a $K_{1,3}$, Egawa and Ota proved the following.

**Theorem 3.11** (*Egawa and Ota* [88]). *Let $k$ be a positive integer. Then if $G$ is a graph of order $4k$ with minimum degree at least $2k$, then $G$ contains a $\mathscr{S}_4$-factor each component of which is isomorphic to a $K_{1,3}$, unless $G$ is isomorphic to $K_{2k,2k}$, with $k$ being odd.*

For an extension to the case where each component of the factor is a $K_{1,t}$, see [118].

Another interesting special case is the following. Let $K_4 - e$ denote the complete graph $K_4$ with one edge $e$ removed.

**Theorem 3.12** (*Kawarabayashi* [208]). *Let $G$ be a graph of order $4k$ with $\delta(G) \geqslant 5k/2$. Then $G$ contains a $(K_4 - e)$-factor.*

The next result specifies the number of components of the factor sought as well as the maximum degrees within the components.

**Theorem 3.13** (*Lovász* [249]). *Let $G$ be a graph with maximum degree $\Delta$ and let $\alpha$ be a non-negative integer. Suppose further that $k_1 + k_2 + \cdots + k_\alpha = \Delta - \alpha + 1$, where each $k_i$ is a non-negative integer. Then $V(G)$ can be partitioned into $\alpha$ disjoint subsets $V_i$ such that each vertex of $G_i = G[V_i]$ is joined to at most $k_i$ other vertices of $G_i$, for $i = 1, \ldots, \alpha$.*

The above result of Lovász has been used to derive best known error bounds in certain branches of coding theory [70].

The next variation, proved independently by Györi [131] and Lovász [252], answered a conjecture of Frank concerning the partitioning the vertex set of the graph into vertex-disjoint subgraphs each of which contains a prescribed vertex.

**Theorem 3.14** (*Lovász* [252], *Györi* [131]). *Let $G$ be a $k$-connected graph and suppose $v_1, \ldots, v_k$ are $k$ distinct vertices of $G$. Suppose further that $|V(G)| = n = n_1 + \cdots + n_k$ is a partition of $|V(G)| = n$ into $k$ positive parts.*

*Then $V(G)$ can be partitioned into $k$ disjoint subsets $V_i$ such that $v_i \in V_i$, $|V_i| = n_i$ and $G[V_i]$ is connected for every $1 \leqslant i \leqslant n$.*

In the next result a minimum degree condition replaces the connectivity condition. (See also [82].)

**Theorem 3.15** (*Enomoto and Matsunaga* [99]). *Let $G$ be a graph of order $n$ and suppose $n = a_1 + \cdots + a_k$ is a partition of $n$ where each $a_i \geqslant 2$. Suppose $\delta(G) \geqslant 3k - 2$. Then given any $k$ distinct vertices $v_1, \ldots, v_k \in V(G)$, $V(G)$ can be partitioned as $V(G) = A_1 \cup \cdots \cup A_k$ such that $|A_i| = a_i$, $v_i \in A_i$ and $\delta(G[A_i]) > 0$, for all $1 \leqslant i \leqslant k$.*

(See [172] and [80] for recent work on closely related problems.)

Certain connectivity or minimum degree conditions on the parent graph may suffice to guarantee a partition the component subgraphs of which have either prescribed connectivity or minimum degree.

**Theorem 3.16.** (a) (*Thomassen [343]*) *For each pair of positive integers* $(s, t)$, *there exist positive integers* $f(s, t)$ *and* $g(s, t)$ *such that each graph* $G$ *with* $\kappa(G) \geqslant f(s, t)$ (*respectively*, $\delta(G) \geqslant g(s, t)$) *admits a partition of its vertex set* $V(G) = S \cup T$ *such that the induced subgraphs* $G[S]$ *and* $G[T]$ *have connectivity* (*respectively*, *minimum degree*) *at least* $s$ *and* $t$, *respectively*.

(b) (*Hajnal [133]*) *Moreover*, *if* $s \geqslant 3$ *and* $t \geqslant 2$, *then* $f(s, t) \leqslant 4s + 4t - 13$ *and if* $s \geqslant 4$, *then* $g(s, t) \leqslant t + 2s - 3$.

(The reader is also referred to [96,98].)

Finally, Nishimura [295] has found that one can determine the existence in a graph $G$ of a factor each component of which is isomorphic to a second graph $H$ by checking certain subgraphs of $G$ for such a factor.

**Theorem 3.17.** *Let* $G$ *and* $K$ *be connected graphs such that* $|V(G)| = n|V(K)|$, *for some* $n \geqslant 2$, *and let* $p$ *be a fixed integer satisfying* $1 < p < n$. *Then if* $G - A$ *has a* $K$-*factor for every connected subgraph* $A$ *with* $|V(A)| = p|V(K)|$, *it follows that* $G$ *also has a* $K$-*factor*.

## 4. Factors in random graphs

There are several popular models of so-called *random* graphs. We will be content to refer to only one of these.

Let $1, \ldots, n$ be a labelling of the vertices and let $\{e_{ij}\}$, $1 \leqslant i < j \leqslant n$, be an array of independent random variables, where each $e_{ij}$ assumes the value 1 with probability $p$ and 0 with probability $1 - p$. This array determines a random graph on $\{1, \ldots, n\}$ where each $(ij)$ is an edge if and only if $e_{ij} = 1$. This probability space (or *random graph*) is denoted by $G_{n,p}$.

An event $E$ concerning a graph $G \in G_{n,p}$ is said to hold *asymptotically almost surely* (or *a.a.s.*), if $\lim_{n\to\infty} \text{Prob } E = 1$.

**Theorem 4.1** (*Erdős and Rényi [104]*). *Let* $n$ *be even and* $p = (1/n)(\log n + w(n))$, *with* $\lim_{n\to\infty} w(n) = \infty$. *Then* $G \in G_{n,p}$ *has a* 1-*factor a.a.s.*

**Theorem 4.2** (*Shamir and Upfal [331]*). *Let* $p = (1/n)(\log n + (r - 1) \log \log n + w(n))$, *with* $r \geqslant 1$ *and suppose* $\lim_{n\to\infty} w(n) = \infty$. *Suppose further that* $f$ *is a mapping from* $V(G)$ *into* $\{1, \ldots, r\}$ *with* $\sum_{i=1}^{r} f(x_i)$ *even. Then* $G \in G_{n,p}$ *has an* $f$-*factor a.a.s.*

For excellent treatments of random graphs and their factors, the reader is referred to [37–41,64,115,174,175,195,196, 276–278,319,322,331,332].

## 5. Graph factorization

Recall that a factorization of a graph $G$ normally refers to a partition of the edge set of $G$ into factors.

### 5.1. 1-factorizations

We begin with the widely studied 1-*factorization Conjecture.* (See [58,363, Ch. 19].)

**Conjecture 5.1.** Let $G$ be a simple graph of even order $n$. If $G$ is regular with $\Delta(G) \geqslant n/2$, then $\chi'(G) = \Delta(G)$; that is, $G$ is Class 1 (i.e., $G$ has a 1-factorization).

To date the best result toward this conjecture is the following due to Chetwynd and Hilton [59] and, independently, to Niessen and Volkmann [289].

**Theorem 5.2.** *If one replaces* $n/2$ *by* $(\sqrt{7} - 1)n/2$, *the above conjecture becomes true*.

See also [51].

The following, due to Plantholt and Tipnis, may be viewed as an extension of the Chetwynd–Hilton–Niessen–Volkmann result to the multigraph case. (See also [308].)

**Theorem 5.3** (*Planthold and Tipnis [307]*). *Let $G$ be a regular multigraph of even order $n$ and multiplicity $\mu(G) \leqslant r$. Then if $\Delta(G) \geqslant r(5n/6+1)$, $\chi'(G) = \Delta(G)$.*

Evidence in favor of the truth of the 1-factorization Conjecture for "large" graphs is given by the following (See [302] and Häggkvist (unpublished)).

**Theorem 5.4.** *Given $\varepsilon > 0$, there is a number $N = N(\varepsilon)$ such that if $G$ is a $\Delta$-regular simple graph of even order greater than $N$, and $\Delta \geqslant (\frac{1}{2}+\varepsilon)|V(G)|$, then $G$ has a 1-factorization.*

Another approach to the conjecture is represented by the following result.

**Theorem 5.5** (*Zhang and Zhu [376]*). *Every $k$-regular graph of order $2n$ contains at least $\lfloor k/2 \rfloor$ edge-disjoint 1-factors, if $k \geqslant n$.*

For general references to edge coloring see [109,108,154,158,176]. (A more general conjecture, called the *overfull conjecture* and due also to Chetwynd and Hilton [58], would imply the 1-factorization Conjecture. See also [153,156].)

### 5.2. $[a, b]$-factorizations

Next we turn to an example of a more general type of factorization, called $[a, b]$-factorization. A graph $G$ is an $[a, b]$-*graph* if $a \leqslant \deg(v) \leqslant b$, for every vertex $v \in V(G)$. A graph then has an $[a, b]$-*factorization* if it has a factorization into an edge-disjoint union of $[a, b]$-graphs each of which spans $G$.

**Theorem 5.6** (*Kano [184]*). *Suppose $a$ and $b$ are integers such that $0 \leqslant a \leqslant b$.*

(i) *A graph $G$ has a $[2a, 2b]$-factorization if and only if $G$ is a $[2am, 2bm]$-graph for some integer $m$, and*
(ii) *every $[8m+2k, 10m+2k]$-graph has a $[1, 2]$ factorization.*

(See also [257] and [13].)

Kano [185] has also obtained the next result on $[a, b]$-factorization which involves the "odd cycle property" introduced in Section 2.1.

**Theorem 5.7.** *Let $a$ and $b$ be integers with $0 \leqslant a \leqslant b$ and let $G$ be a graph with the odd cycle property. Then $G$ is $[a, b]$-factorizable if and only if $G$ is an $[an, bn]$-graph for some positive integer $n$.*

We next present a sample result on $(g, f)$-factorization; that is, a representation of $E(G)$ as the edge-disjoint union of $(g, f)$-factors. (See [369,370] for other such results.)

**Theorem 5.8** (*Yan et al. [371]*). *Let $G$ be a multigraph and let $g$ and $f$ be two functions mapping $V(G)$ into the non-negative integers. Let $m$ be a positive integer and $\ell$ an integer with $0 \leqslant \ell \leqslant 3$ and $\ell \equiv m \pmod 4$. Then if $G$ is an $(m + 2\lfloor m/4 \rfloor + \ell, mf - 2\lfloor m/4 \rfloor - \ell)$ graph, $G$ has a $(g, f)$-factorization.*

Era [102] and Egawa [78] seem to have been the first to consider factorizations in which the factors are "almost regular". Two of their results appear below combined into one corollary to the following theorem due, very recently, to Hilton [155].

**Theorem 5.9.** *Let $G$ be a simple $d$-regular graph and suppose $r \geqslant 2$. Then*

(i) *$G$ has an $([r, r+1]$-factorization with exactly $x$ $[r, r+1]$-factors if*
 (1-a) *$d/(r+1) < x < d/r$ or*
 (1-b) *if $r$ is odd and $x = d/(r+1)$ or*
 (1-c) *if $r$ is even and $x = d/r$.*

(ii) *If $r$ is even and $(r+1)|d$, then there are d-regular simple graphs which are, and d-regular simple graphs which are not, $[r, r+1]$-factorizable into $x = d/(r+1)$ $[r, r+1]$-factors; if $r$ is odd and $r|d$, then there are d-regular simple graphs which are, and d-regular simple graphs which are not $[r, r+1]$-factorizable into $x = (d/r)$ $[r, r+1]$-factors.*

(iii) *If $x \notin [d/(r+1), d/r]$, then no d-regular simple graph is $[r, r+1]$-factorizable into $x$ $[r, r+1]$-factors.*

**Corollary 5.10** (*Era [102], Egawa [78]*). *Let $k \geqslant 2$ be an integer. Then*:

(i) *every r-regular graph $G$ with $r \geqslant 4k^2$ has a $[2k, 2k+1]$-factorization and*
(ii) *every $(k^2 - 4k + 2)$-regular graph $G$ has a $[2k-1, 2k]$-factorization.*

Lonc [248] has shown that if $\mathscr{S}^-$ is a family of stars from which the one and two edge stars are deleted, it is *NP*-complete to decide whether a bipartite graph admits an $\mathscr{S}^-$ factorization.

Some observations about connections between the 1-factorization problem and combinatorial designs may be found in [138]. The subjects of 1-factorizations of graphs in general, or complete graphs in particular, are enormous topics unto themselves and quickly lead one into the discipline of combinatorial design theory. We will not treat these topics further, but instead direct the interested reader to several excellent surveys [333,273,239,274] and the more recent encyclopedic volume of Wallis [363].

### *5.3. Linear arboricity*

A different type of edge partition problem is represented by the concept of *linear arboricity*. The *linear arboricity* $\ell a(G)$, of graph $G$ is the minimum number of paths which together partition $E(G)$. (See [136]. An alternate name for this parameter is *path-chromatic index*. See [151].)

Akiyama, Exoo and Harary showed the following.

**Theorem 5.11** (*Akiyama et al. [10]*). *If $G$ is any graph, $\ell a(G) \leqslant \lceil 3\lceil \Delta/2\rceil/2\rceil$.*

There are two interesting conjectures on linear arboricity in existence. The first one was formulated in [9], the second in [151] and both remain unsettled.

**Theorem 5.12** (*Linear arboricity conjecture I (LAC-I)*). *The linear arboricity of every d-regular graph is $\lceil (d+1)/2\rceil$.*

**Theorem 5.13** (*Linear arboricity conjecture II (LAC-II)*). *The linear arboricity of any graph $G$ is bounded above by $\lceil (\Delta(G)+1)/2\rceil$.*

McDiarmid and Reed [271] proved that for every $d$, almost all $n$-vertex, $d$-regular graphs satisfy LAC-I and later [272] showed that almost all graphs satisfy LAC-II. Guldan [128] proved LAC-I when the degree of regularity is large with respect to the order of the graph. Truszczynski [350] proved that if LAC-I holds for each of two regular graphs $G$ and $H$, then it holds for their Cartesian product $G \times H$. Wu [367] proved LAC-II for all planar graphs with $\Delta \neq 7$. Alon [14] proved LAC-II for every graph with even (respectively, odd) maximum $\Delta$ and girth at least $50\Delta$ (respectively, $100\Delta$). In the same paper, Alon also proved that LAC-II is asymptotically correct in the sense that the linear arboricity is bounded above by $(1/2+\varepsilon)\Delta$ for every graph $G$ with sufficiently large $\Delta$. Péroche [303] showed that it is *NP*-complete to determine whether it is possible to partition the edge set $E(G)$ into $k$ paths when $\Delta(G) = 2k$.

There has also been work done on the related parameter called *k-linear arboricity*, denoted $\ell a(G)$, which is defined to be the minimum number of paths, each of length no more than $k$, which partition $E(G)$. See [132,19]. Note that $\ell_1(G)$ is just the edge chromatic number of graph $G$.

### *5.4. Factorizations of complete graphs*

The problem of partitioning the edge set of a complete graph $K_n$ into disjoint factors has received widespread attention. The existence of such a factorization has been studied for many different prescribed properties of the factors involved.

One interesting example arises when one limits the diameter of the factors. The first research upon this problem dates from Bosák et al. [45]. For $k \geqslant 2$, let $f(k)$ denote the smallest positive integer such that the complete graph of order $f(k)$ admits a factorization into $k$ factors, each having diameter 2. Znam [379] proved that if $k$ is sufficiently large, then $f(k)=6k$. Bosák [43] had shown earlier that $6k-52 \leqslant f(k) \leqslant 6k$ for all values of $k \geqslant 2$. Scattered results involving bounds for small values of $k$ exist (e.g., see [260]), but a complete determination of $f(k)$ remains unsettled.

The variant of the above problem obtained when one prescribes the radii of the factors, rather than the diameters, seems to have been first proposed by Rosa. See [301,346] for results on this variation and when both the diameter and radius of the factor is prescribed, see [285,347,348]. If it is demanded that the factors of given diameter must also be isomorphic, see [222,150]. If the number of factors of given diameter is prescribed, see [45,299,300,169,117].

Several interesting new problems concerned with factoring the complete graph into factors all of which are $r$-regular are posed in [157] and the authors make some progress in several special cases.

Martin [261,262] has studied the factorization of the complete bipartite graph $K_{m,n}$ into factors of type $K_{p,q}$.

## 6. Factor algorithms and complexity

Throughout this section, let $n=|V(G)|$ and $m=|E(G)|$. The first polynomial algorithm for matching in an arbitrary graph was formulated by Edmonds [76] and has come to be known as the *blossom algorithm*. Its running time is $O(n^4)$. The fastest algorithm to date for maximum matching in a general (i.e., not necessarily bipartite) graph has complexity $O(m\sqrt{n})$ and is due to Micali and Vazirani [275]. (Curiously, a proof of correctness of this algorithm was not published until fourteen years later! (See [361]. See also [305].)) Since the Micali–Vazirani algorithm was introduced, two other matching algorithms [122,35] having the same complexity as Micali–Vazirani have been produced.

Faster algorithms exist, however, in certain special cases. If the graph is 3-regular and has no cutedge, then by the classical result of Petersen [304], the graph must have a 1-factor. For this case, an $O(n \log^4 n)$ algorithm is given in [34] for finding a 1-factor. If, in addition, the graph is planar, then an $O(n)$ algorithm is also given.

The Gabow, Kaplan and Tarjan algorithm mentioned above [121] can be modified to test whether a graph has a *unique* $f$-factor and find it, if it exists, and to check whether a given $f$-factor is unique, all in polynomial time.

Anstee [24] gave algorithmic proofs of both the $(g, f)$-factor theorem and the $f$-factor theorem and his algorithms either return one of the factors in question or show that none exists, all in $O(n^3)$ time. Note that this complexity bound is independent of the functions $g$ and $f$.

$F_c$-factors (introduced in Section 3) can be determined in polynomial time.

**Theorem 6.1** (*Mühlbacher [279], Hell and Kirkpatrick [143]*). *There is a polynomial algorithm for finding an $F_c$-factor or showing that none exists.*

A polynomial algorithm for finding a 2-factor, if one exists, was first found by Edmonds and Johnson [77]. If one additionally demands that the 2-factor be triangle-free, the problem remains polynomially solvable. (See [67,139].)

For graphs in general, if one demands that the forbidden cycle lengths form a non-empty subset of $\{5, 6, \ldots\}$, the problem has been shown to be NP-complete [147]. If one forbids only $C_3$, $C_4$ and $C_5$, the problem is again NP-complete [Papadimitriou; see 67]. The complexity in the two remaining cases, namely where only 4-cycles are forbidden or where only triangles and 4-cycles are forbidden, remains unresolved. For the intermediate case when the graph is bipartite and it is required to find a 2-factor which has no 4-cycle component, there is a polynomial algorithm [140]. (For an extension of Hartvigsen's result to $f$-factors, see [211].)

The problem of deciding whether or not a graph has a Hamilton cycle was one of first decision problems proved to be *NP*-complete by Karp [197,198]. The problem remains *NP*-complete, even if the graphs are restricted to be 3-regular and planar [125] or 4- or 5-regular and planar [306]. These results would seem to indicate that few *connected* factor problems are likely to be polynomially solvable.

Let $G$ be an arbitrary graph. A *G-factor* of a graph $H$ is a set $\{G_1, \ldots, G_d\}$ of subgraphs of $H$ such that each $G_i$ is isomorphic to $G$ and the sets $V(G_i)$ collectively partition $V(G)$.

Let $FACT(G)$ denote the recognition problem:

INSTANCE: A graph $H$.

QUESTION: Does $H$ admit a $G$-factor?

The answer to $FACT(K_1)$ is (trivially) always "yes" and so $FACT(K_1) \in P$. Problem $FACT(K_2)$ is just the question of the existence of a perfect matching in $H$ and hence also lies in $P$. More generally, if $G$ consists of a disjoint union of copies of $K_1$ and $K_2$, $FACT(G)$ belongs to $P$. Interestingly, however, we have the next result [213]. (See [212] for a nice application to exam scheduling and see also [144].)

**Theorem 6.2.** *If some component of G has more than two vertices, then $FACT(G)$ is NP-complete.*

A set $\mathscr{S}$ (respectively, $\mathscr{S}_n$) of stars is said to be *sequential* if $\mathscr{S} = \{K_{1,1}, K_{1,2}, K_{1,3}, \ldots\}$ (respectively $\mathscr{S}_n = \{K_{1,1}, K_{1,2}, K_{1,3}, \ldots, K_{1,n}\}$). For the $\mathscr{S}$-factor and $\mathscr{S}_n$-factor problems mentioned earlier, Hell and Kirkpatrick have shown the following.

**Theorem 6.3** (*Hell and Kirkpatrick [146]*). *The problem of finding a star factor $\mathscr{S}$ (or $\mathscr{S}_n$) in a graph is polynomial if and only if the set of stars is sequential.*

The *clique partition number* of a graph $G$ is the smallest number $cp(G)$ such that there exists a set of $cp(G)$ cliques in $G$ such that the cliques form a partition of $E(G)$. A graph $G$ is *chordal* if every cycle in $G$ of length greater than 3 has a chord.

**Theorem 6.4** (*Ma et al. [254]*). *The problem of determining $cp(G)$ is NP-hard, for the class of $K_4$-free graphs and for the class of chordal graphs. However, the problem is polynomial for the class of graphs which are both $K_4$-free and chordal.*

That the problem of determining the chromatic index of a graph is *NP*-complete was first proved by Holyer [166]. If $G$ is bipartite however, a classical theorem of König states that

**Theorem 6.5** (*König [216,217]*). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

König's proof yields an $O(mn)$ algorithm to produce an optimal edge coloring. However, recently faster algorithms have been invented. Presently, it seems that either an algorithm of Kapoor and Rizzi [194] or an algorithm of Schrijver [328] is best, depending upon the relative sizes of $|V(G)|$ and $\Delta(G)$. If the bipartite graphs involved are, in addition, regular, even faster algorithms exist. (See [318].)

Holyer also determined the complexity of the problem of partitioning $E(G)$ into complete graphs *all of the same order greater than* 2. (Compare this problem with the clique partition problem stated above.)

**Theorem 6.6** (*Holyer [167]*). *Suppose $n \geqslant 3$. Then the problem of partitioning $E(G)$ into copies of $K_n$ is NP-complete.*

Holyer used the above result to prove five other edge partition problems to be *NP*-complete in the same paper.

A still more general type of edge partition problem is the *H*-decomposition Problem.

*The H-decomposition Problem*: given a fixed graph $H$, can the edge set of an input graph $G$ be partitioned into copies of $H$?

A beautiful result of Gustavsson [130] (see also [15]) says that if $G$ is "dense enough and large enough" (as a function of $H$), then such an $H$-decomposition is always possible. Let $gcd(G)$ denote the greatest common divisor of the degrees of $G$.

**Theorem 6.7.** *Let H be a graph with h edges. There exist constants $N_0 = N_0(H)$ and $\gamma = \gamma(H) > 0$, such that for all $n > N_0$, if G is a graph on n vertices and m edges, with $\delta(G) \geqslant n(1-\gamma)$, $gcd(H)|gcd(G)$, and $h|m$, then G has an H-decomposition.*

The Holyer complexity results mentioned above deal with special cases of the *H*-decomposition problem. More recently, Dor and Tarsi have proved the following.

**Theorem 6.8** (*Dor and Tarsi [75]*). *The H-decomposition problem is NP-complete whenever H contains a connected component with three edges or more.*

Bryś and Lonc [50] have finished off the complexity issue here by showing that, in all other cases, the $H$-decomposition Problem is polynomial.

If both the graph $G$ to be factored and the components of the factors are of some special types, sometimes factorization can be guaranteed and even accomplished in polynomial time. An example is the following.

**Theorem 6.9** (*Bertram and Horák [33]*). *There is a polynomial algorithm which finds a factorization of any given* 4-*regular graph into two triangle-free* 2-*factors or else shows that such a factorization does not exist*.

On the other hand, the same two authors pose the following:

**Conjecture 6.10.** The problems of

(a) recognizing which $2n$-regular graphs decompose into two triangle-free $n$-factors, and
(b) recognizing which $2n$-regular graphs decompose into $n$ triangle-free 2-factors are both $NP$-complete for all $n \geqslant 3$.

The following conjecture would follow from the truth of the 1-factorization conjecture.

**Conjecture 6.11** (*Hilton [152]*). Let $G$ be a $d$-regular simple graph of order $2n$ and let $d = p_1 + \cdots + p_r$ be a partition of $d$. If $d \geqslant n$, then $G$ has a factorization into edge-disjoint subgraphs $H_1 \cup \cdots \cup H_r$, where $H_i$ is regular of degree $p_i$.

The author proves the conjecture true in various special cases.

What if the subgraphs which partition $E(G)$ are all to be *isomorphic*? Again, it is instructive to compare this partition problem with the $H$-decomposition discussed above. Of course the difference is that here it is not specified ahead of time what the isomorphic factor graphs are, but it is only specified that they be isomorphic.

If graph $G$ admits a partition of its edge set into $t$ isomorphic subgraphs, then we say that *$G$ is divisible by $t$*. Of course an obvious necessary condition for $G$ to be divisible by $t$ is that the number of graphs in the partition must divide $|E(G)|$.

Following [91,92], let us call a graph $G$ *$t$-rational* if either $G$ is divisible by $t$ or else $t \not\| E(G)|$.

*The t-rational problem*: Given a graph $G$ and a positive integer $t$, is $G$ $t$-rational?

In [366], it is shown that if $r > 2t$, then almost all labelled $r$-regular graphs cannot be factorized into $t \geqslant 2$ isomorphic subgraphs. But curiously, no examples of such regular non-factorizable graphs are known which satisfy the obvious necessary divisibility condition: $t||EG)|$.

We provide several other sample results.

**Theorem 6.12** (*Ellingham and Wormald [94]*). *Let $G$ be a multigraph and suppose $t$ is an integer such that $t \geqslant \chi'(G)$. Then $G$ is $t$-rational*.

Thus, by Vizing's theorem, we have the next result.

**Theorem 6.13.** *If $G$ is $r$-regular and $t \geqslant r + 1$, then $G$ is $t$-rational*.

It was shown by Harary et al. [137], and independently by Schönheim and Bialostocki [326], that $K_n$ is $t$-rational and Wang [364] and Quinn [316] independently showed that $K_{n,n,\ldots,n}$ is $t$-rational.

In [94] it is shown that every 3-regular graph can be partitioned into three isomorphic subgraphs; in [91] that every 4-regular graph can be partitioned into four such subgraphs and in [92] it is shown that for $r$ even and $r = 25, 27$ or $r \geqslant 29$, every $r$-regular graph can be partitioned into $r$ such subgraphs.

To the best of our knowledge, the computational complexity of the $t$-rational problem is currently unknown.

Plesník [309] has studied the complexity of the prescribed diameter and radius decompositions of the complete graph discussed at the end of the preceding section. The *diameter decomposition problem* is to decompose the edge set of a given graph into $k$ disjoint factors with given diameters $d_1, d_2, \ldots, d_k$. If radii instead of diameters are prescribed, then the corresponding problem is called the *radius decomposition problem*.

**Theorem 6.14.** (1) *The diameter decomposition problem for graphs is NP-hard even in the case of two factors with diameter bound* 2;

(2) *the diameter decomposition problem is NP-hard even for bipartite graphs in the case of two factors with diameter bound* 3;

(3) *the radius decomposition problem for graphs is NP-hard even in the case of two factors with radius bound* 2; *and*

(4) *the diameter decomposition problem for digraphs is NP-hard even in the case of symmetric digraphs and two factors with diameter bound* 2.

## 7. Subgraph problems

Another variation on the theme of factors is represented by the following collection of problems. Suppose $G$ and $H$ are two graphs on $p$ vertices. When is $H$ isomorphic to a subgraph of $G$? A sufficient condition involving minimum degree $\delta$, maximum degree $\Delta$ and independence number $\alpha$ was given by Catlin [52].

**Theorem 7.1.** *Let $G$ and $H$ be two graphs on $p$ vertices. If*

$$\delta(G) \geqslant p - \frac{\alpha(H)}{2\Delta(H)} - 1,$$

*then $H$ is isomorphic to a subgraph of $G$.*

A different sort of result along this line was proved by Erdős and Hajnal [103].

**Theorem 7.2.** *Given positive integers $n$ and $k$, there exists a positive integer $N$ such that for every integer $n > N$, for every graph $G$ of order $n$ containing neither a clique of order $\lfloor c \log n \rfloor$ nor $\lfloor c \log n \rfloor$ independent vertices, and for every graph $H$ of order $k$ we have that $H$ is isomorphic to an induced subgraph of $G$.*

It is trivial to embed any $k$-regular graph $G$ in a $(k+1)$-regular graph, if we omit the requirement that the $k$-regular graph *span* the $(k+1)$-regular graph. Just form the Cartesian product of $G$ with a single edge. But what if one wants to minimize the number of extra vertices needed? Let us denote by $v(G)$ the minimum number of extra vertices needed.

**Theorem 7.3** (*Gardiner [123]*). *Let $B$ be $k$-regular and have order $n$. Then*

(i) *if $\overline{G}$ has a* 1-*factor*, $v(G)=0$,
(ii) *if $\overline{G}$ has no* 1-*factor and $n$ and $k$ are of opposite parity*, $v(G)=1$, *while*
(iii) *if $\overline{G}$ has no* 1-*factor and $n$ and $k$ are of the same parity, then $n<2k$ and $v(G)=k+2$.*

Given a graph $G$, does it contain a $k$-regular subgraph? (Here again we do not require that the $k$-regular subgraph span $G$.) This is sometimes referred to as the *k-regular subgraph recognition problem*. If $k=1$ or 2, clearly the problem is polynomial. The $k$-regular subgraph recognition problem has gained popularity due in large part to a conjecture of Berge which was proved in [341,377].

**Theorem 7.4.** *Every* 4-*regular simple graph contains a* 3-*regular subgraph*.

It is interesting to note that these proofs do not provide an algorithm for finding the 3-regular subgraph.

More recently, it has been shown [62,310,53] that the $k$-regular subgraph recognition problem is, in fact, *NP*-complete for all $k \geqslant 3$. If the $k$-regular subgraph sought is, in particular, complete, then there is an $O(n^k)$ algorithm [282] to solve the problem, where $n=|V(G)|$.

A different result involving 3-regular subgraphs is given next. (See [16,17].) The proof uses number-theoretic methods.

**Theorem 7.5.** *If $G$ is a multigraph in which all vertices have degree $k$ or $k+1$ and at least one vertex has degree at least* 5, *then $G$ contains a* 3-*regular subgraph*.

A different type of subgraph problem is the subject of the next result (see [87]).

**Theorem 7.6.** *If G is a graph with* $|V(G)| \geqslant 4k+6$ *and* $\delta(G) \geqslant k+2$, *then G contains k pairwise vertex-disjoint copies of* $K_{1,3}$.

Let us at least mention another large family of problems closely allied to factor and factorization problems, namely so-called *packing* problems. Here instead of searching for a factor of a particular kind in a given graph $G$, one seeks a subgraph of $G$ of maximum order which admits a given factor. Hence, in a sense, packing problems generalize factor problems. The problem of packing certain restricted graph families into other graphs has been studied widely. For example, packing edges and triangles [66], cliques and maximal cliques [66,213], complete bipartite subgraphs [146], $P_k$-matchings (that is perfect 2-matchings in which all cycles of length $\leqslant k$ are forbidden) [68], fractional matchings [357,279,30] and dynamic matchings [298]. (See [65].) Again, space dictates that we refrain from pursuing this tack, but instead we refer the interested reader to [247] for a nice survey of the state of the art. (But let the reader be warned that the term "packing" means different things to different authors!)

This is just a tip of the iceberg. There are hundreds of papers in the literature dealing with a wide variety of "subgraph problems" as well as "graph decompositions". Space constraints dictate that we dare not venture further in these directions. Instead the reader is referred to the survey papers [31,60,73,74,142,321] and the books [44,72,63].

## 8. Uncited references

[18,162].

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