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SPANNING SUBGRAPHS WITH SPECIFIED VALENCIES

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Abstract. The author has published a necessary and sufficient condition for a finite loopless graph to have a spanning subgraph with a specified positive valency at each vertex (see [8, 9]). In the present paper it is contended that the condition can be made more useful as a tool of graph theory by imposing a maximality condition.

1. The condition for an f-factor

Let G be a finite graph. Loops and multiple joins are allowed. Let f be a function from the vertex-set $V(G)$ of G to the set of non-negative integers. We define an *f-factor* of G as a spanning subgraph F of G such that the valency of x in F is $f(x)$ for each vertex x of G. We recall that the "valency" of a vertex x in a graph is the number of incident edges, loops being counted twice.

Let us define a *G-triple* as an ordered triple (S, T, U), where S, T and U are disjoint subsets of $V(G)$ whose union is $V(G)$.

Let x be a vertex of G, and Y a subset of $V(G)$. If x is in Y, we define $\lambda(Y, x)$ as the number of links joining x to vertices in $Y \setminus \{x\}$, plus twice the number of loops incident with x. But if x is not in Y, we define $\lambda(Y, x)$ as the number of links joining x to vertices in Y.

Let Y be any subset of $V(G)$. Consider the subgraph of G induced by Y, that is, consisting of the vertices of Y, the loops on these vertices and the links with both ends in Y. We refer to the components of this subgraph simply as the "components of Y".

Let $B = (S, T, U)$ be a G-triple. We describe a component C of U as *odd* or *even* (with respect to B) according as the number

$$(1) \qquad J(C) = \sum_{b \in V(C)} \{f(b) + \lambda(T, b)\}$$

is odd or even. We denote the number of odd components of U by $h(B)$. We also write

$$(2) \qquad \delta(B) = h(B) - \sum_{a \in S} f(a) + \sum_{c \in T} \{f(c) - \lambda(T, c) - \lambda(U, c)\} .$$

We call B an *f-barrier* if $\delta(B)$ is positive.

The main theorem of [8] and [9] can be stated as follows.

Theorem 1. *Let G be finite and loopless, and let $f(x)$ be positive for each vertex x. Then G has either an f-factor or an f-barrier, but not both.*

If a graph G has an f-barrier, it has a *maximal* f-barrier, that is, an f-barrier B with the greatest value of $\delta(B)$ consistent with the given G and f.

It has been found in practice that it is difficult to apply Theorem 1 to the solution of theoretical problems about f-factors. It appears, however, from the results of the present paper that most of the difficulty is inessential, and can be avoided by using maximal f-barriers instead of arbitrary ones.

2. Transfers of vertices

Let $B = (S, T, U)$ be any G-triple. If x is a vertex of S, we define $\mu(x)$ as the number of odd components C of U such that some link of G joins x to a vertex of C. If x is in U, we define $\mu(x)$ in the same way, but in terms of the triple $(S \cup \{x\}, T, U \setminus \{x\})$.

Suppose x is in S. We consider the change in $\delta(B)$ when x is transferred to U, and B is accordingly transformed into $B_1 = (S \setminus \{x\}, T, U \cup \{x\})$.

We observe that $\mu(x)$ of the odd components of U in B, together perhaps with some of the even components, are replaced by a single component K of $U \cup \{x\}$ in B_1. The component K is odd or even in B_1 according as

$$\mu(x) + f(x) + \lambda(T, x)$$

is odd or even. The remaining components of U in B persist as components of $U \cup \{x\}$ in B_1, with the same parities.

We deduce that

$$h(B_1) - h(B) = -\mu(x) + \eta(x) , \tag{3}$$

where $\eta(x)$ is 0 or 1, with the parity of $\mu(x) + f(x) + \lambda(T, x)$. Moreover, when B is replaced by B_1, the terms

$$-\sum_{a \in S} f(a) \quad \text{and} \quad \sum_{c \in T} \{f(c) - \lambda(T, c) - \lambda(U, c)\}$$

of (2) increase by $f(x)$ and $-\lambda(T, x)$, respectively. We deduce that

$$\delta(B_1) - \delta(B) = -\mu(x) + \eta(x) - \lambda(T, x) + f(x) . \tag{4}$$

If B is a maximal f-barrier, the difference $\delta(B_1) - \delta(B)$ must be nonpositive. We thus have

2.1. *If $B = (S, T, U)$ is a maximal f-barrier and if $x \in S$, then*

$$f(x) \leq \mu(x) + \lambda(T, x) - \eta(x) . \tag{5}$$

We note that $\mu(x)$ and $\eta(x)$ have the same values for B_1 as for B. We apply (4) with B_1 a maximal f-barrier, and then interchange the symbols B and B_1. We thus deduce

2.2. *If $B = (S, T, U)$ is a maximal f-barrier and if $x \in U$, then*

$$f(x) \geq \mu(x) + \lambda(T, x) - \eta(x) . \tag{6}$$

If $f(x) = \mu(x) + \lambda(T, x) - \eta(x)$, we say that x is a *left-neutral* vertex of B, whether it belongs to S or to U. Applying (4) we deduce

2.3. *Let $B = (S, T, U)$ be a maximal f-barrier of G, and let x be a left-*

neutral vertex in B. Then B remains a maximal f-barrier of G when x is transferred from one of the sets S and U to the other.

In each of the inequalities (5) and (6) the two sides agree in parity, by the definition of $\eta(x)$.

We go on to give a closely analogous theory of the transfer of vertices between T and U. If $x \in T$, we define $\nu(x)$ as the number of odd components C of U such that some edge of G joins x to a vertex of C. If $x \in U$, we define $\nu(x)$ in the same way, but in terms of the triple $(S, T \cup \{x\}, U \setminus \{x\})$.

Suppose x is in T. We consider the change in $\delta(B)$ when x is transferred to U, and B is accordingly transformed into $B_2 = (S, T \setminus \{x\}, U \cup \{x\})$.

We note that $\nu(x)$ of the odd components of U in B, together perhaps with some of the even components, are replaced by a single component Q of $U \cup \{x\}$ in B_2, the one including the vertex x. We find that Q is odd or even in B_2 according as

$$\nu(x) + f(x) + \lambda(T, x) + \lambda(U, x)$$

is odd or even. The remaining components of U in B persist as components of $U \cup \{x\}$ in B_2, with the same parities.

We deduce that

$$h(B_2) - h(B) = -\nu(x) + \xi(x)\,, \tag{7}$$

where $\xi(x)$ is 0 or 1, with the parity of

$$\nu(x) + f(x) + \lambda(T, x) + \lambda(U, x)\,.$$

Moreover, when B is replaced by B_2, the terms

$$-\sum_{a \in S} f(a) \quad \text{and} \quad \sum_{c \in T} \{f(c) - \lambda(T, c) - \lambda(U, c)\}$$

of (2) increase by 0 and $\lambda(T, x) + \lambda(U, x) - f(x)$, respectively. We deduce that

(8) $$\delta(B_2) - \delta(B) = -\nu(x) + \xi(x) + \lambda(T, x) + \lambda(U, x) - f(x) .$$

If B is a maximal f-barrier, the difference on the left must be non-positive. We deduce

2.4. *If $B = (S, T, U)$ is a maximal f-barrier and if $x \in T$, then*

(9) $$f(x) \geq \lambda(T, x) + \lambda(U, x) - \nu(x) + \xi(x) .$$

We note that $\nu(x)$, $\xi(x)$ and $\lambda(T, x) + \lambda(U, x)$ have the same values for B_2 as for B. We apply (8) with B_2 a maximal f-barrier, and then interchange the symbols B and B_2. We find

2.5. *If $B = (S, T, U)$ is a maximal f-barrier and if $x \in U$, then*

(10) $$f(x) \leq \lambda(T, x) + \lambda(U, x) - \nu(x) + \xi(x) .$$

If

$$f(x) = \lambda(T, x) + \lambda(U, x) - \nu(x) + \xi(x) ,$$

then whether x belongs to T or to U we say that it is a *right-neutral* vertex of B. Applying (8), we obtain

2.6. *Let $B = (S, T, U)$ be a maximal f-barrier of G, and let x be a right-neutral vertex in B. Then B remains a maximal f-barrier of G when x is transferred from one of the sets T and U to the other.*

In each of the inequalities (9) and (10) the two sides agree in parity, by the definition of $\xi(x)$.

3. Small values of $f(x)$

In this section we derive some elementary consequences of Propositions 2.1–2.6.

3.1. *If G has an f-barrier, it has a maximal f-barrier $B = (S, T, U)$ such that $x \in S \cup U$ whenever $f(x) \leq 1$.*

Proof. G has a maximal f-barrier (S, T, U). By transferring right-neutral vertices in T to U, we can arrange for the strict inequality to hold in 2.4. The expression on the right of (9) cannot be negative since $\nu(x)$ is, by definition, not greater than $\lambda(U, x)$. But if it is zero, then $f(x)$ must be even, by the definition of $\xi(x)$. Hence, by the strict inequality, $f(x)$ must be at least 2 for each $x \in T$.

3.2. *If G has an f-barrier, it has a maximal f-barrier $B = (S, T, U)$ such that $x \in S$ whenever $f(x) = 0$.*

Proof. By 3.1, there is a maximal f-barrier $B = (S, T, U)$ of G such that $x \in S \cup U$ whenever $f(x) = 0$. By transferring left-neutral vertices in U to S, we arrange for the strict inequality to hold in 2.2. Now the right side of (6) can be negative only if $\mu(x) = \lambda(T, x) = 0$ and $\eta(x) = 1$. But then $f(x)$ is odd and so at least 1, by the definition of $\eta(x)$. Hence, by the strict inequality, $f(x)$ must be at least 1 for each $x \in U$.

4. Some slight generalizations of Theorem 1

As an exercise in the foregoing theory we show how to generalize Theorem 1 to any non-negative f and any finite graph G.

4.1. *Let G be a loopless finite graph, and let f be any function from $V(G)$ to the set of non-negative integers. Then G has an f-factor or an f-barrier, but not both.*

Proof. Let Z be the set of all vertices x of G such that $f(x) = 0$. Let G' be the graph obtained from G by deleting the vertices of Z and all their incident edges. Let f' be the function from $V(G')$ induced by f. It follows from (2) that if $B' = (S', T', U')$ is a G'-triple, and B is the G-triple $(Z \cup S', T', U')$, then $\delta(B) = \delta(B')$, where $\delta(B)$ is defined in terms of f and $\delta(B')$ in terms of f'. Thus B is an f-barrier of G if and only if B' is an f'-barrier of G'.

By Theorem 1, G' has an f'-factor or an f'-barrier, but not both. But an f'-factor of G' clearly determines an f-factor of G, and conversely. On the other hand if G' has an f'-barrier (S', T', U'), then G has an f-

barrier $(Z \cup S', T', U')$. Finally, if G has an f-barrier, it has a maximal f-barrier $B = (S, T, U)$ such that $Z \subseteq S$, by 3.2. Then $(S \setminus Z, T, U)$ is an f'-barrier of G'. This completes the proof.

Theorem 1'. *Let G be any finite graph, and let f be any function from V(G) to the set of non-negative integers. Then G has either an f-factor or an f-barrier, but not both.*

Proof. Let us enumerate the loops of G as $s_1, s_2, ..., s_n$. We write v_j for the vertex of G incident with s_j.

We construct from G a loopless graph G' as follows: We introduce $2n$ new vertices, two vertices p_j and q_j for each loop s_j of G. For each loop s_j, we introduce three new edges $p_j v_j$, $q_j v_j$ and $p_j q_j$, and replace s_j by the triangle $v_j p_j q_j$. Let f' be the function from $V(G')$ to the set of non-negative integers such that $f'(v) = f(v)$ if v is a vertex of G, and $f'(p_j) = f'(q_j) = 1$ for each relevant suffix j. We write Z for the set of the $2n$ new vertices p_i, q_j.

Consider any G-triple (S, T, U). Let B' be the G'-triple $(S, T, U \cup Z)$. If v_j is in S or T, then p_j and q_j are the only vertices of one even component of $U \cup Z$ in B'. The components of U in B persist as components of $U \cup Z$ in B', with their loops replaced by triangles but with no alteration in parity. It follows from (2) that $\delta(B') = \delta(B)$. Thus B' is an f'-barrier of G' if and only if B is an f-barrier of G.

By 4.1, G' has an f'-factor or an f'-barrier, but not both.

An f-factor F of G gives rise to an f'-factor F' of G' when each loop s_j occurring in F is replaced by the two edges $v_j p_j$ and $v_j q_j$, and the edge $p_k q_k$ is adjoined for each loop v_k of G not occurring in F. It is clear moreover that each f'-factor of G' can be obtained from some f-factor F of G in this way. Thus G has an f-factor if and only if G' has an f'-factor.

If G has an f-barrier (S, T, U), then G' has the f'-barrier $(S, T, U \cup Z)$. On the other hand if G' has an f'-barrier (S', T', U'), we can suppose it maximal, with $Z \subseteq S' \cup U'$, by 3.1. Suppose, however, that p_j is in S' for some loop s_j of G. Then $\mu(p_j)$ is at most 1, and so is $\lambda(T', p_j)$. Hence p_j is left-neutral, by 2.1. Similarly, q_j is left-neutral if it is in S'. We can therefore choose the maximal f'-barrier (S', T', U') so that $Z \subseteq U'$, by 2.3. But then $(S', T', U' \setminus Z)$ is an f-barrier of G. Thus G has an f-barrier if and only if G' has an f'-barrier.

This completes the proof of the theorem.

In applications of Theorem 1′ it is well to bear in mind that the numbers $f(x)$, $x \in V(G)$, must sum to an even number if G is to have an f-factor. If they sum to an odd number, G has the f-barrier $(\emptyset, \emptyset, V(G))$.

5. 1-factors

If $f(x) = 1$ for each $x \in V(G)$, we refer to an f-factor of G as a *1-factor* of G. A necessary and sufficient condition for the existence of a 1-factor of G can be stated as follows.

Theorem 2. *A finite graph G is without a 1-factor if and only if there is a subset S of $V(G)$ such that*

$$(11) \qquad |S| < h(S)\,,$$

where $|S|$ is the number of elements of S and $h(S)$ is the number of components of $V(G) \setminus S$ having an odd number of vertices.

This $h(S)$ should be distinguished from the $h(B)$ defined in Section 1. However, if $f = 1$ and $B = (S, T, U)$, where T is null, we find by comparing definitions that $h(B) = h(S)$.

Theorem 2 is readily deduced from Theorem 1 or 1′, with the auxiliary Propositions 2.1–2.6. The distinction between the looped and the loopless cases is utterly trivial for Theorem 2. Using 3.1 we find that G is without a 1-factor if and only if it has a maximal 1-barrier $B = (S, T, U)$ in which T is null. The assertion that $\delta(B) > 0$ is equivalent to (11).

Whether the above argument is to be counted as a proof of Theorem 2 depends on whether we regard Theorem 2 as part of the proof of Theorem 1. The proof of Theorem 1 in [8] is constructive and does not depend on Theorem 2, but the proof given in [9], supposed to be shorter, derives Theorem 1 from Theorem 2. Direct proofs of Theorem 2 can be found in [7] and [4] (see also [6]).

The present theory is not constructive since we have not given an algorithm for finding a maximal f-barrier when some f-barrier is given. But we can construct an adequate substitute, satisfying 2.1 to 2.6, by transferring vertices one at a time so as to increase $\delta(B)$, or leave it unchanged, at each step until no further increase in $\delta(B)$ is found possible. Reference may be made to [2] for a discussion of constructive methods in the theory of subgraphs with specified valencies.

6. A theorem of Berge

In this and the next section we try to demonstrate the utility of Theorem 1′ by exhibiting some well-known theorems as simple consequences of it. The first of these concerns "matchings".

A *matching* of a finite graph G can be defined as a subgraph H of G in which each vertex has valency 1. There may, however, be vertices of G that do not belong to H. We refer to the difference $|V(G)| - |V(H)|$ as the *deficiency* of the matching. Thus a 1-factor of G is a matching with deficiency zero.

C. Berge has given a generalization of Theorem 2 that can be stated as follows (see [1, p. 154]).

Theorem 3. *Let G be a finite graph, and let d be a non-negative integer. Then in order that G shall have no matching of deficiency d, it is necessary and sufficient that one of the following three conditions shall hold:*

(i) $d > |V(G)|$,

(ii) $d + |V(G)|$ *is odd,*

(iii) *there is a subset S of $V(G)$ such that $d + |S| < h(S)$.*

Proof. We construct from G a graph G' by adjoining a single new vertex w and then joining w to each vertex of G by a single new link. We now write $f(x) = 1$ if $x \in V(G)$, and $f(w) = d$. Evidently, G has no matching of deficiency d if and only if G' has no f-factor, that is, if and only if G' has an f-barrier $B = (Z, T, U)$, by Theorem 1′.

If G' has such an f-barrier, we may suppose that B is maximal and that $T \subseteq \{w\}$, by 3.1.

Suppose $T = \{w\}$. Then every component of U in B is even. Hence $d = f(w) > |V(G)|$, by (2). This (i) holds. Conversely, if this condition holds, G' has the f-barrier $(V(G), \{w\}, \emptyset)$.

In the remaining case we can suppose T null. If w is in U, then $h(B)$ is either 0 or 1. It then follows from (2) that Z is null and $h(B) = 1$. Thus G' is itself an odd component of $U = V(G')$. Accordingly, (ii) holds. Conversely, if (ii) holds, G' has the f-barrier $(\emptyset, V(G'), \emptyset)$.

In the remaining case T is null and w is in Z. Write $S = Z \setminus \{w\}$. Then, by (2), S satisfies (iii). Conversely, if some subset S of $V(G)$ satisfies this condition, then G' has the f-barrier $(S \cup \{w\}, V(G) \setminus S, \emptyset)$.

7. A theorem of Erdös and Gallai

A *strict* graph is a graph without loops or multiple joins. An example is K_d, the d-clique or complete d-graph, which has d vertices, no loops, and exactly one edge joining each pair of distinct vertices.

Let $(f_1, f_2, ..., f_p)$ be a partition of the positive even integer $2q$ into p parts $f_1 \geq f_2 \geq ... \geq f_p$. We call this partition P *strictly graphic* if there is a strict graph G of d vertices such that the numbers f_i are the valencies of the vertices of G. We may ask under what conditions is a given partition P of $2q$ strictly graphic.

It is easy to put this problem into a form to which Theorem 1′ is applicable. Let the vertices of K_p be enumerated as $v_1, v_2, ..., v_p$. Write $f(v_j) = f_j$ for each vertex v_j. Then the partition $\mathrm{P} = (f_1, f_2, ..., f_p)$ is strictly graphic if and only if K_p has an f-factor.

P. Erdös and T. Gallai have given the following theorem (see [3]; also see [5, p. 59]).

Theorem 4. P *is strictly graphic if and only if*

$$\sum_{i=1}^{r} f_i \leq r(r-1) + \sum_{i=r+1}^{p} \min(r, f_i) \tag{12}$$

for each integer r satisfying $1 \leq r \leq p - 1$.

We proceed to prove this in terms of Theorem 1′.

Proof. By Theorem 1′ (or Theorem 1 if the f_i are all non-zero), P is not strictly graphic if and only if K_p has an f-barrier $B = (S, T, U)$. Evidently, $h(B) \leq 1$ for any such B.

If such a B exists, we can suppose it maximal. Using (5) and (6) we then find that $f(x) \leq |T| + 1$ if $x \in S$, and that $f(x) \geq |T| - 1$ if $x \in U$, with equality possible in each case only if x is left-neutral. Hence, by 2.3, we can arrange that $f(x) \leq |T|$ if $x \in S$, and $f(x) \geq |T|$ if $x \in U$.

Writing $|T| = r$ and using (2) we now deduce from the condition $\delta(B) > 0$ that

$$\sum_{i=1}^{r} f_i > \sum_{a \in S} f(a) + r(r - 1 + |U|) - h(B) . \tag{13}$$

Moreover, if the two sides of (13) differ only by 1, we can adjust the notation so that T is the set of vertices with suffixes from 1 to r. However, in that case the parity of the difference is that of

$$\sum_{b \in U} f(b) + r|U| - h(B)$$

since the sum of $f(x)$ over all the vertices of G is the even number $2q$. Accordingly, the difference is even, by the definition of an odd component. From this contradiction we conclude that the two sides of (13) differ by at least 2. Hence (13) remains valid as a strict inequality even when the term $-h(B)$ is deleted. This result implies that $1 \leq r \leq p - 1$. It is thus contrary to (12).

Conversely, if (12) fails for some r, we consider the K_p-triple $B = (S, T, U)$, where T consists of the r vertices with suffixes 1 to r and S consists of all remaining vertices a such that $f(a) < r$. Then (13) holds. It follows that B is an f-barrier of K_p. Accordingly, P is not strictly graphic.

This completes the proof of the theorem.

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