Independence of the Dual Axiom in Modal K with Primitive $\Diamond$

Richmond H. Thomason
Philosophy Department
University of Michigan

`rthomaso@umich.edu`
`http://www.eecs.umich.edu/~rthomaso/`

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Abstract

Explicit axioms relating $\Diamond\phi$ and $\Box\phi$ appear to be needed if $\Diamond$ is taken to be primitive. It is proved here that in fact such axioms are indispensable.

1. Introduction

In [Blackburn *et al.*, 2001], systems of propositional modal logic are formulated with $\Diamond$ as primitive. In axiomatizing these logics, they resort to an axiom that is not needed when $\Box$ is the modal primitive. This is the axiom *Dual*:

$$\Diamond p \to \neg\Box\neg p.$$

The purpose of this paper is to show that such an axiom is indispensable: $\Diamond p \to \Diamond\neg\neg p$ and $\Diamond\neg\neg p \to p$ can be invalidated in a modal logic with $\Diamond$ as primitive and with the usual boolean axioms, the necessitation rule, and the **K** axiom. Of course, these axioms can't be invalidated in Kripke frames, or even in boolean propositional logic. So the models used in this paper are somewhat exotic.

2. Eight-valued models for modality

We will use many-valued models with 8 values. It's best to think of these values as made up out of two 4-element boolean algebras $\mathbf{B}$ and $\mathbf{B}'$. Here is a picture.

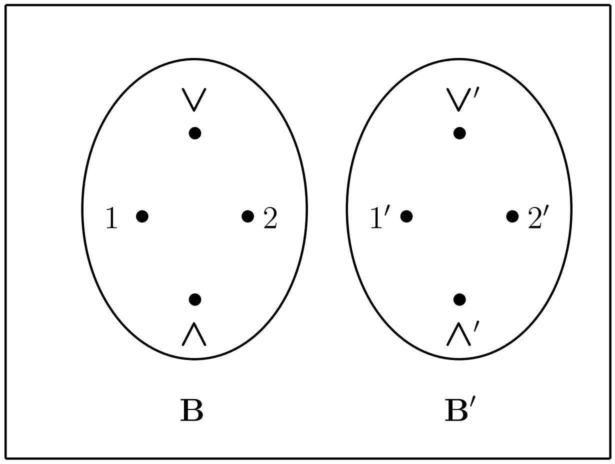

Figure 1: An 8-valued model

$\bigvee$ and $\bigvee'$, the units of the two boolean algebras, are the only designated values: a formula is valid if it only receives values in $\{\bigvee, \bigvee'\}$. Negation is nonstandard. Within $\mathbf{B}$, it is as expected, but the "complement" of an element of $\mathbf{B}'$ is the complement of its twin in $\mathbf{B}$. The conditional $\to$ is the "union" or least upper bound of the complement of the antecedent with the twin of the consequent.

We now spell these ideas out explicitly in the following definition of the functions $f_\neg$ and $f_\to$ that serve to interpret negation and the conditional.

Definition 1. $f_\neg$, $f_\to$.

$$
\begin{array}{l}
f_\neg(\bigwedge) = \bigvee,\ \ f_\neg(\bigvee) = \bigwedge,\ \ f_\neg(1) = 2,\ \ f_\neg(2) = 1. \\
f_\neg(\bigwedge') = \bigvee,\ \ f_\neg(\bigvee') = \bigwedge,\ \ f_\neg(1') = 2,\ \ f_\neg(2') = 1. \\
\mathrm{Twin}(x) = f_\neg(f_\neg(x)) \ \ = x \text{ if } x \in \mathbf{B},\ \ f_\neg(f_\neg(x)) \text{ if } x \in \mathbf{B}' \\
\text{For } x, y \in \mathbf{B}:\ \ f_\to(\bigwedge, x) = \bigvee,\ \ f_\to(x, \bigwedge) = f_\neg(x),\ \ f_\to(\bigvee, x) = x, \\
\qquad\qquad\quad f_\to(1,2) = 2,\ \ f_\to(2,1) = 1. \\
\text{For } x, y \in \mathbf{B}':\ \ f_\to(x, y) = f_\to(\mathrm{Twin}(x), \mathrm{Twin}(y)).
\end{array}
$$

These conditions overlap in places, but the overlaps are consistent.

We will postpone the definition of $f_\Diamond$.

3. Axioms

The system in which we are interested has the following four axioms, together with the rules of *modus ponens*, necessitation, and substitution.

1. $p \to_\bullet q \to p$
2. $(p \to_\bullet q \to r) \to_\bullet (p \to q) \to_\bullet p \to r$
3. $(\neg p \to \neg q) \to_\bullet q \to p$
4. $\neg\Diamond\neg(p \to q) \to_\bullet \neg\Diamond\neg p \to \neg\Diamond\neg q$

Axioms 1–3 are complete for boolean propositional logic. Axiom 4 is the modal axiom **K**, with $\Diamond$ primitive. Together, these axioms partially axiomatize the modal system **K**, including all the usual axioms, but not the Dual Axiom.

4. Preliminaries

The relation $x \preceq y$ over $\mathbf{B} \cup \mathbf{B}'$ is given by least upper bound in Figure 2. It is the transitive closure of

$$\{\langle \bigwedge, 1\rangle, \langle \bigwedge, 2\rangle, \langle 1, \bigvee\rangle, \langle 2, \bigvee\rangle, \langle \bigwedge', 1'\rangle, \langle \bigwedge', 2'\rangle, \langle 1', \bigvee'\rangle, \langle 2', \bigvee'\rangle\}.$$

We begin with some easily verifiable claims, stated without proof.

Claim 1. $f_{\rightarrow}(x, y) \in \{\bigvee, \bigvee'\}$ iff $f_{\rightarrow}(x, y) = \bigvee$.

Claim 2. $f_{\rightarrow}(x, y) = \bigvee$ iff $x \preceq y$ iff $\text{Twin}(x) \preceq \text{Twin}(y)$.

Claim 3. $f_{\neg}(x) \preceq f_{\neg}(y)$ iff $y \preceq x$ iff $\text{Twin}(y) \preceq \text{Twin}(x)$.

Claim 4. Where *lub* is the least upper bound operator in $\mathbf{B} \cup \mathbf{B}'$,
$f_{\rightarrow}(x, y) = lub(f_{\neg}(x), \text{Twin}(y))$.

Claim 5. Where *glb* is the least upper bound operator in $\mathbf{B} \cup \mathbf{B}'$, $f_{\rightarrow}(x, f_{\rightarrow}(y, z)) = \bigwedge$ iff $glb(\text{Twin}(x), \text{Twin}(y)) \preceq \text{Twin}(z)$.

Claim 6. $f_{\rightarrow}(w, f_{\rightarrow}(x, f_{\rightarrow}(y, z))) = \bigvee$ iff $glb(\text{Twin}(w), \text{Twin}(x), \text{Twin}(y)) \preceq \text{Twin}(z)$.

5. Nonmodal soundness

Let V be a mapping of propositional varaibles to values in $\mathbf{B} \cup \mathbf{B}'$. V is extended to nonmodal formulas in the usual way, interpreting $\neg$ with $f_{\neg}$ and $\rightarrow$ with $f_{\rightarrow}$.

The validity of Axioms 1–3 and the rules of substitution and *modus ponens* follows from the fact that $V(\phi) = V'(\phi)$ if $V'(\phi) = V(\text{Twin}(\phi))$. But I'll also provide direct arguments.

Validity of the substitution rule. Substitution is valid in any many-valued matrix.

Validity of *modus ponens*. Suppose that $V(\phi \rightarrow \psi), V(\phi) \in \{\bigvee, \bigvee'\}$ and let $\text{Twin}(V(\psi)) = x$. By Claims 1 and 2, $\bigvee \preceq x$, so $x = \bigvee$, so $V(\psi) \in \{\bigvee, \bigvee'\}$.

Validity of Axiom 1. Suppose that $V(p \rightarrow_{\bullet} q \rightarrow p) \notin \{\bigvee, \bigvee'\}$. By Claim 1, and Claim 5, $glb(x, y) \not\preceq x$, where $x = \text{Twin}(V(p)), y = \text{Twin}(V(q))$. But this is impossible.

Validity of Axiom 2. Consider $(p \rightarrow_{\bullet} q \rightarrow r) \rightarrow_{\bullet} (p \rightarrow q) \rightarrow_{\bullet} p \rightarrow r$, and let $x = \text{Twin}(V(p)), y = \text{Twin}(V(q)), z = \text{Twin}(V(r))$.

Suppose now that $V((p \rightarrow_{\bullet} q \rightarrow r) \rightarrow_{\bullet} (p \rightarrow q) \rightarrow_{\bullet} p \rightarrow r) \notin \{\bigvee, \bigvee'\}$, Then, by Claims 1 and 6 and the definition of $f_{\rightarrow}$, $glb(f_{\rightarrow}(x, f_{\rightarrow}(y, z)), f_{\rightarrow}(x, y), x) \not\preceq z$. Let GLB be $glb(f_{\rightarrow}(x, f_{\rightarrow}(y, z)), f_{\rightarrow}(x, y), x)$. Then either:

- i. $z = \bigwedge$ and $GLB \in \{\bigvee, 1, 2\}$, or
- ii. $z = 1$ and $GLB \in \{\bigvee, 2\}$, or
- iii. $z = 2$ and $GLB \in \{\bigvee, 1\}$.

In Case i, $GLB = glb(f_{\to}(x, f_{\neg}(y)), f_{\to}(x, y), x) = \bigwedge$, and we have a contradiction, because $\bigwedge \preceq z$ and, by hypothesis, $GLB \not\preceq z$.

In Case ii, in view of the definition of $f_{\to}$, either:

ii.1 $f_{\to}(y, 1) = 1$, or
ii.2 $f_{\to}(y, 1) = \bigwedge$.

In Case ii.1, $GLB = glb(f_{\to}(x, 1), f_{\to}(x, y), x)$ and $y \in \{\bigwedge, 2\}$. If $y = \bigwedge$, $GLB = glb(f_{\to}(x, 1), f_{\neg}(x), x) \in \{\bigwedge, 1\}$ and we have a contradiction. If $y = 2$, $GLB = glb(f_{\to}(x, 1), f_{\to}(x, 2), x) = \bigwedge$ and again we have a contradiction.

In Case ii.2, $GLB = glb(f_{\to}(x, \bigwedge), f_{\to}(x, y), x) = glb(f_{\to}(x, y), x)$ and $y \in \{1, \bigwedge\}$. If $y = 1$, $GLB = glb(f_{\to}(x, 1), x)$ and $y \in \{1, \bigwedge\}$. If $y = 1$, $GLB = glb(f_{\to}(x, 1), x) \in \{1, \bigwedge\}$ and we have a contradiction. If $y = \bigwedge$, $GLB = glb(f_{\neg}(x), x) = \bigwedge$ and again we have a contradiction.

The reasoning in Case iii is like that in Case ii.

Validity of Axiom 3. By Claim 4, $f_{\to}(f_{\neg}(x), f_{\neg}(y)) = lub(f_{\neg}(f_{\neg}(x)), \mathrm{Twin}(f_{\neg}(y))) = lub(\mathrm{Twin}(x), f_{\neg}(y))$. But $f_{\to}(y, x) = lub(f_{\neg}(y), \mathrm{Twin}(x))$. Therefore $V(\neg p \to \neg q) = V(q \to p)$, so that $V(\neg p \to \neg q) \to (q \to p) = \bigvee$.

This completes the detailed proof of soundness for the boolean axioms.

6. Interpreting $\Diamond$

To interpret $\Diamond$, we revert to the picture of our model in Figure 2, and elaborate it by including two regions of $\mathbf{B}$ and $\mathbf{B}'$. These are shown by the dashed lines in the following elaborated picture.

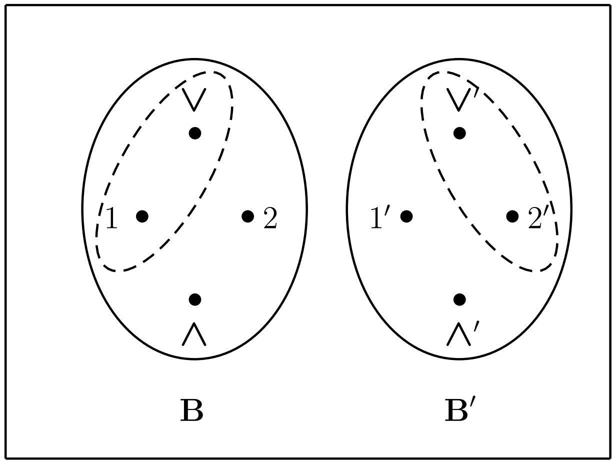

Figure 2: Regions of the eight-valued model

The interpretation of $\Diamond$ is sensitive to whether you are working in $\mathbf{B}$ or in $\mathbf{B}'$. In the former case, the value is $\bigvee$ for arguments in the area $\{1, \bigvee\}$. Otherwise it is $\bigwedge$. In the latter case, the value is $\bigvee$ for arguments in the circled area $\{2', \bigvee'\}$. Otherwise it is $\bigwedge$. Here is the official definition.

For $x \in \mathbf{B}$, $f_\Diamond(x) = \bigvee$ if $x \in \{1, \bigvee\}$ and $f_\Diamond(x) = \bigwedge$ if $x \notin \{1, \bigvee\}$.
For $x \in \mathbf{B}'$, $f_\Diamond(x) = \bigvee$ if $x \in \{2', \bigvee'\}$ and $f_\Diamond(x) = \bigwedge$ if $x \notin \{2', \bigvee'\}$.

7. Modal soundness

Let $f_\Box(x) = f_\neg(f_\Diamond(f_\neg(x)))$. We state two more easily verified claims.

Claim 7. For all x, $f_\Box(x) \in \{\bigvee, \bigvee'\}$.

Claim 8. For all x, $f_\Box(x) = \bigvee$ iff $\mathrm{Twin}(x) \in \{\bigvee, 1\}$.

We now check the validity of the necessitation rule. Suppose that for all V, $V(\phi) \in \{\bigvee, \bigvee'\}$. In view of Claim 8. then, $f_\Box(x) = \bigvee$, where $x = \mathrm{Twin}(V(\phi))$. So $V(\neg\Diamond\neg\phi) = \bigvee$, for all V.

Now consider Axiom 4: $\neg\Diamond\neg(p \to q) \to_\bullet \neg\Diamond\neg p \to \neg\Diamond\neg q$. Let $\mathrm{Twin}(V(p)) = x$, $\mathrm{Twin}(V(q)) = y$, and suppose that $lub(f_\Box(f_\to(x, y)), f_\Box(x)) \not\preceq f_\Box(y)$. Then, in view of Claim 7, $f_\Box(f_\to(x, y)) = \bigvee$, $f_\Box(x) = \bigvee$, $f_\Box(y) = \bigwedge$. By Claim 8, $f_\to(x, y) \in \{\bigvee, 1\}$, $f_\Box(x) = \{\bigvee, 1\}$, and $f_\Box(y) = \{\bigwedge, 2\}$. But this is impossible.

This completes the proof of the soundness of the boolean and modal axioms and rules for this interpretation. It remains to be shown that Dual is invalid.

8. Invalidity of Dual

Recall that the Dual Axiom is $\Diamond p \to \neg\Box\neg p$. Since with $\Diamond$ primitive, '$\Box p$' is defined as '$\neg\Diamond\neg p$', and $f_\neg(f_\neg(x)) = x$, this axiom amounts to $\Diamond p \to \Diamond\neg\neg p$.

To invalidate Dual, let $V(p) = 2'$. Then $V(\Diamond p) = \bigvee$. But $V(\neg\neg p) = 1$, so $V(\Diamond\neg\neg p) = \bigwedge$. So $V(\Diamond p \to \Diamond\neg\neg p) = \bigwedge$. $\Diamond\neg\neg p \to \Diamond p$ is invalid as well. Let $V(p) = 1'$. Then $f_\Diamond(p) = \bigwedge$ and $f_\Diamond(f_\neg(f_\neg(p))) = f_\Diamond(1) = \bigwedge$.

This completes the proof of the independence of Dual from the other axioms.

9. Conclusion

It would be nice if the models used in this proof were useful for some other purpose, but none has yet occurred to me.

Bibliography

[Blackburn *et al.*, 2001] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, Cambridge, England, 2001.