Abstraction in First-Order Modal Logic[1]

by

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1. When an abstraction operator is added to the classical (i.e., extensional) predicate calculus with identity, no essential change is effected in the logic. New formulas such as $\hat{x}P(x)\,(a)$ are added to the language of the logic, but in view of equivalences such as

(1) $\hat{x}P(x)\,(a) \equiv P(a)$,

any formula involving abstracts is equivalent to a formula free of these operators. (*Note*: we assume here that abstracts are not regarded as singular terms: $Q(\hat{x}P(x))$ is not a formula, from our present point of view.) In modal calculi, however, it is by no means clear that this is the case, since if $\Box P$ is substituted for P in (1), the resulting equivalence involves a change of scope; a is within the scope of the necessity-sign in $\Box P(a)$, but not in $\hat{x}\Box P(x)\,(a)$. In fact, philosophers have often made much of the distinction between assertions such as

(2) Necessarily, the President of the U.S. is a citizen of the U.S.

and

(3) The President of the U.S. is necessarily a citizen of the U.S.

The first amounts, roughly, to "It is necessarily the case that any President of the U.S. is a citizen of the U.S." But the second says, "the person who in fact is the President of the U.S. has the property of necessarily being a citizen of the U.S." Thus, while (2) is clearly true, it would be reasonable to consider (3) false.

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Such a distinction has struck many as unclear or nonsensical; using abstracts, however, it is easy to render in formal notation the difference between (2) and (3). It is natural to formalize (2) as

$$(4)\quad \Box Q(\iota_x P(x))$$

or, alternatively, as

$$(5)\quad \Box(\hat{x}Q(x)\,(\iota_x P(x))),$$

whereas (3) is rendered by

$$(6)\quad \hat{x}\Box Q(x)\,(\iota_x P(x)).$$

Of course, the mere syntactical difference between (4) and (6) is not sufficient to establish that they represent different assertions. And indeed, in view of the principle

$$(7)\quad A^t/x \equiv \hat{x}A(t)$$

of abstraction, one would expect (4) and (6) to be logically equivalent. In general, however, we should beware in modal logic of accepting principles which hold for classical logic, without first devising some semantical criterion for distinguishing those which hold in the more general case from those which do not; and (7) is a good case in point. We will show by extending a familiar semantical interpretation of modal logic to include abstracts that it is quite plausible to regard (7) as *invalid* when modal operators are involved in A. Moreover, we will show that with respect to this interpretation (4) and (6) are not equivalent; in particular, (4) and the negation of (6) are simultaneously satisfiable. ((4) and (5), however, which *do* seem to be equivalent formulations of (2), turn out to be equivalent with respect to our semantics.)[2]

2. The interpretation on which we will build is the one given in [1] of the modal logic called Q3; this gives an adequate modeling of truth-functions, necessity, quantifiers, identity, and descriptions. We will use the semantical notation of [1] and [2]. Let

[2] These considerations suggest a number of philosophical applications: for instance, in accounting for the traditional distinction between modality *de dicto* and modality *de re*. Applications such as this are discussed in a more philosophical paper, [3].

$\langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle$ be a Q3ms (i.e., a model structure for Q3). Here, $\mathcal{K}$ is the set of possible worlds, $\mathcal{R}$ is the relation of relative possibility on $\mathcal{K}$, $\mathcal{D}$ is a function taking members α of $\mathcal{K}$ into domains D_α of individuals, and $\mathcal{D}'$ is a set of individuals disjoint from $\bigcup_{\alpha \in \mathcal{K}} \mathcal{D}_\alpha$. Finally, D is the union of all these domains; $D = \mathcal{D} \cup \bigcup_{\alpha \in \mathcal{K}} \mathcal{D}_\alpha$. Our extended logic, say Q3^r, differs from Q3 in having additional atomic formulas of the kind $\hat{x}A(t)$, where x is an individual variable, A a formula, and t an individual term.[3] To generalize the semantical interpretation of Q3 to this logic, we must provide a rule of satisfaction for such formulas. This is accomplished as follows.

$$I_\alpha(\hat{x}A(t)) = I^{I_\alpha(t)}/x_\alpha(A).$$

In other words, $\hat{x}A(t)$ is true in α in case the thing referred to by t in α satisfies A in α.

Under this interpretation, the implication

$$(7)\quad \Box P(\iota_x P(x)) \supset \hat{x}\Box P(x)\,(\iota_x P(x))$$

is invalid, as is shown by the following countermodel. Let $\mathcal{K} = \{\alpha, \beta\}$, $\mathcal{D}_\alpha = \mathcal{D}_\beta = \{1, 2\}$, $\mathcal{D}' = \{3\}$, and $\alpha\mathcal{R}\alpha$, $\alpha\mathcal{R}\beta$, and $\beta\mathcal{R}\beta$. Let $I_\alpha(P) = \{1\}$ and $I_\beta(P) = \{2\}$. Then $I_\alpha(\iota_x P(x)) = 1$ and $I_\beta(\iota_x P(x)) = 2$, so that $I_\alpha(P(\iota_x P(x)) = T$ and $I_\beta(P(\iota_x P(x)) = T$; hence $I_\alpha(\Box P(\iota_x P(x)) = T$. On the other hand, $I_\alpha(\iota_x P(x)) = 1$, and $I^1/x_\beta(P(x)) = F$; hence, $I^{I}\alpha(\iota_x P(x))/x_\alpha(\Box P(x)) = F$, so that $I_\alpha(\hat{x}\Box P(x)\,(\iota_x P(x)) = F$.

Thus, (7) is invalid. Moreover, within the context of our semantical theory we can see that it is invalid for reasons which parallel the traditional distinction. Necessarily, the President of the U.S. is a President of the U.S., but it is not the case that the man who is President of the U.S. is necessarily a President of the U.S.

3. Our modeling of Q3^r gives rise in the usual way to notions of simultaneous satisfiability, implication, and validity, and it is natural to ask how these notions may be formulated deductively. In view of the semantical completeness results of [1], this question

[3] Abstracts with n argument-places are definable as follows: $\hat{x}_1\hat{x}_2 \ldots \hat{x}_n A(t_1, \ldots, t_n) =\mathrm{df}\ \hat{x}_1\hat{x}_2 \ldots \hat{x}_n A(t_n) \ldots (t_2)\,(t_1)$.

can be settled by determining what axioms must be added to those of Q3 in order to capture our extended notion of validity.

This problem can be settled rather simply. Let $Q3^r$ have, besides all the axioms and rules of Q3, all axioms of the following kind.

$$\text{(Ab) } x = t \supset (A^x/y \equiv \hat{y}A(t))$$

It is readily verified that any formula falling under this scheme is valid; indeed, suppose $I_\alpha(x=t)=T$. Then $I_\alpha(x)=I_\alpha(t)$, so that $I^{I_\alpha(x)/y_\alpha}(A)=I^{I_\alpha(t)/y_\alpha}(A)$. But $I_\alpha(A^x/y)=I^{I_\alpha(t)/y_\alpha}(A)$, and $I^{I_\alpha(t)/y_\alpha}(A)=I_\alpha(\hat{y}A(t))$. Thus, $I_\alpha(A^x/y)=I_\alpha(\hat{y}A(t))$, so that $I_\alpha(A^x/y \equiv \hat{y}A(t))=T$. It follows that $x=t\supset(A^x/\hat{y}\equiv\hat{y}A(t))$ is valid.

4. Fortunately, all of the difficulties involved in proving $Q3^r$ semantically complete have been resolved in showing that Q3 is complete; the proof given in [1] needs only to be augmented in a straightforward way to yield the result that if a set Γ of formulas is $Q3^r$-consistent, then Γ is simultaneously $Q3^r$-satisfiable.

First, the definition in [1] of Q3-*M*-saturation is changed in the natural way; a $Q3^r$-*M*-saturated set Γ of formulas is a set of formulas of *M* which is $Q3^r$-consistent, and which meets the other requirements of Q3-saturation. The proofs given in [1] then show that any $Q3^r$-consistent set of formulas of *M* has a $Q3^r$-M'-saturated extension, where M' is an ω-extension of *M*; and that if Γ is an *M*-saturated set and $\Diamond A \in \Gamma$, then there is an *M*-saturated extension of $\{\Box B/B\in\Gamma\}\cup\{A\}$.

The only remaining task in establishing completeness is to show that if $\underline{\Gamma}$ is *M*-$Q3^r$-saturated then $\underline{\Gamma}$ is simultaneously $Q3^r$-satisfiable. Here, one constructs a $Q3^r$ms $\langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle$ just as in the proof of *L16* in [1]; $\mathcal{K}$, e.g., is $\{\underline{\Delta}/\underline{\Delta}$ is $Q3^r$-*M*-saturated and $\underline{\Gamma}\subset\underline{\Delta}\}$. An interpretation I is then defined on this $Q3^r$ms as in [1], and an inductive argument used to show that for all formulas *A* of *M*, for all $\underline{\Delta}\in\mathcal{K}$, $A\in\underline{\Delta}$, if and only if $I_{\underline{\Delta}}(A)=T$. The only case of this argument which is not presented in [1] is the one novel to $Q3^r$, in which *A* is $\hat{y}B(t)$. By the construction of $\underline{\Delta}$, there is an individual variable *x* of *M* such that $x=t\in\underline{\Delta}$. And I is arranged so that $I_{\underline{\Delta}}(t)=I_{\underline{\Delta}}(x)$. We then argue as follows.

By the axiom (Ab), $\hat{y}B(t)\in\underline{\Delta}$ if and only if $B^x/y\in\underline{\Delta}$. By the hypothesis of induction, this if and only if $I_{\underline{\Delta}}(B^x/y)=T$, and by a lemma concerning substitution, this if and only if $I^{I_{\underline{\Delta}}(x)/y_{\underline{\Delta}}}(B)=T$. But since $I_{\underline{\Delta}}(t)=I_{\underline{\Delta}}(x)$, this if and only if $I^{I_{\underline{\Delta}}(t)/y_{\underline{\Delta}}}(B)=T$; and finally, by the definition of satisfaction for abstracts, this if and only if $I_{\underline{\Delta}}(\hat{y}B(t))=T$.

This establishes that $\hat{y}B(t)\in\underline{\Delta}$ if and only if $I_{\underline{\Delta}}(\hat{y}B(t))=T$, as desired. It follows by induction that, in particular, $\underline{\Gamma}=\{A/I_{\underline{\Gamma}}(A)==T\}$.

Since every $Q3^r$-consistent set is extendible to a $Q3^r$-saturated set, and every $Q3^r$-saturated set has been shown to be simultaneously $Q3^r$-satisfiable, it follows that every $Q3^r$-consistent set is simultaneously $Q3^r$-satisfiable. Conversely, it is easy to show that every simultaneously $Q3^r$-satisfiable set is $Q3^r$-consistent. Thus, we have at last the following completeness results for $Q3^r$.

> A set Γ of formulas of $Q3^r$ is $Q3^r$-consistent if and only if it is simultaneously $Q3^r$-satisfiable.
>
> A formula *A* of $Q3^r$ is $Q3^r$-provable if and only if it is $Q3^r$-valid.

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