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PHILOSOPHICAL PROBLEMS IN LOGIC

Some Recent Developments

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SOME COMPLETENESS RESULTS FOR MODAL PREDICATE CALCULI*

I. INTRODUCTION

Two systems of first-order predicate calculus with identity, one of them with definite descriptions, will be formulated in this paper along with semantic interpretations, and then shown strongly complete by methods similar to those of Henkin [3]. These systems, **Q1** and **Q3**,[1] are generalizations of the systems presented in Kripke [6] and [8], respectively.[2] An informal and philosophical account of **Q1** and **Q3** can be found in [11], together with a historical note concerning the development of the systems and their interpretation.

It is known that a semantically complete interpretation of a system with **S4**-type modality also produces a complete interpretation of the corresponding intuitionistic system. This result for intuitionistic logic has already appeared (again following the lead of Kripke, in [9]) in Thomason [12]. In presenting the syntax and semantic interpretation of **Q1** and **Q3** below, I have used the same format as in the last-named article.

The methods of proof which we will use below apply generally to many different sorts of modality: in particular, to alethic and deontic versions of **S4** and **S5**, von Wright's **M**, and the 'Brouwersche' **B**. But to simplify our presentation, we confine ourselves below to systems which involve the modality of alethic **S4**, leaving it to the reader to generalize the arguments to other modalities. See reference 7 in this connection.

II. MORPHOLOGY

A *morphology* **M** for the first-order modal predicate calculus with identity is a structure made up of the following (disjoint) components:

(1) An infinite well-ordered set V_M of objects called *individual variables*;
(2) A well-ordered set C_M of objects called *individual constants*;
(3) For each nonnegative integer i, a well-ordered set P^i_M of objects called *i-ary predicate letters.*

Let $\supset$, $\sim$, $\Box$,), (, and $=$ be objects not in any of the P^i_M, or in V_M or C_M. These seven objects (or *logical symbols*), together with the members of V_M, C_M, and the P^i_M, comprise the symbols of **M**. The set T_M of *terms* of **M** is $C_M \cup V_M$; the set W_M of formulas of **M** consists of certain finite strings of symbols of **M**, and is defined in the usual way. Using the orderings posited in (1)–(3) above, W_M can easily be well-ordered. (*Remark*: we will frequently suppress mention of the morphology in cases where no confusion can result from doing so. We will assume in this paper that for all morphologies **M**, the sets V_M, C_M, P^i_M – and hence the set W_M – are at most denumerable, and are ordered alphabetically by the positive integers. The main results of this paper depend on this assumption; see references 4 and 10 in this connection.)

If **M** is a morphology with identity and descriptions, another symbol, $\imath$, is added and T_M and W_M are defined by simultaneous induction. (The inductive clause for terms is that $\imath_x A \in T_M$ if $A \in W_M$ and $x \in V_M$.)

Where A is a formula and s and t terms, let A^s/t be the result of replacing all free occurrences of t in A by occurrences of s – relettering bound variables, if necessary, to avoid rendering the new occurrences of s bound in A^s/t. And let $A^s//t$ be any result of replacing various (not necessarily all, or even any) free occurrences of t in A by occurrences of s – again, relettering if necessary.

We will use '$A \vee B$', '$A \wedge B$', '$A \equiv B$', '$\Diamond A$', '$(\exists x) A$' and '$A \prec B$' as abbreviations (in our metalanguage) of '$((A \supset B) \supset B)$', '$\sim(A \supset \sim B)$', '$\sim((A \supset B) \supset \sim(B \supset A))$', '$\sim\Box\sim A$', '$\sim(x)\sim A$', and '$\Box(A \supset B)$', respectively. '$Et$' and '$E\Box t$' refer respectively to $(\exists x)\, x = t$ and to $(\exists x)\Box\, x = t$, where x is the alphabetically first individual variable differing from t, and '$(\exists! x)A$' to $(\exists y)(x)(A \equiv x = y)$, where y is the alphabetically first individual variable not to occur in A.

III. DEDUCIBILITY IN THE SYSTEM Q1

In **Q1** a deductive structure is imposed on morphologies **M** for the first-order modal predicate calculus with identity (but without descriptions) by closing a set of axioms under certain rules. The axioms are determined by stipulating that any tautology is an axiom, as well as any instance of the following eight schemes.[3]

A1. $\Box(A \supset B) \supset . \Box A \supset \Box B$
A2. $\Box A \supset A$
A3. $\Box A \supset \Box\Box A$
A4. $(x)A \supset A^t/x$, where t is any term
A5. $s = s$
A6. $s = t \supset . A \supset A^s//t$
A7. $(x)\Box A \supset \Box (x)A$
A8. $\Diamond s = t \supset \Box s = t$.

The rules of proof in **Q1** are as follows.

R1. $\dfrac{A \qquad A \supset B}{B}$

R2. $\dfrac{A}{\Box A}$

R3. $\dfrac{A \supset B}{A \supset (x)B}$ where x is not free in A.

Deducibility in **Q1** may be defined in much the same way as in Montague and Henkin [10]. Call a member A_i of a sequence $A_1, \ldots, A_n$ of formulas **Q1**-*categorical* in that sequence if some subsequence of $A_1, \ldots, A_i$ is a proof in **Q1** of A_i, and let a sequence $B_1, \ldots, B_k$ be a **Q1**-*derivation* of B_k from a set Γ of formulas if for all i, $1 \leqslant i \leqslant k$, B_i is an axiom of **Q1** or a member of Γ, or follows from previous members of the sequence by R1, or is categorical in the sequence by R2 or R3. And '$\Gamma \vdash_1 A$' means that there is a **Q1**-derivation of A from Γ. Finally, Γ is **Q1**-*consistent* if there is a formula A such that it is not the case that $\Gamma \vdash_1 A$.

For reference in proving metatheorems to come, we record the following facts about **Q1**-deducibility.

T1. If $\Gamma \vdash_1 A$ then $\{\Box B / B \in \Gamma\} \vdash_1 \Box A$.
T2. $\vdash_1 s = t \supset \Box s = t$.

IV. Q1-SATURATION

D1. A subset Γ of W_M is **Q1-M**-*saturated* (abbreviated 'M-saturated' in this section and in Sections V and VI, below) if it meets the following three conditions:

(1) Γ is **Q1**-consistent;
(2) For all $A \in W_M$, $A \in \Gamma$ or $\sim A \in \Gamma$;
(3) For all $A \in W_M$ and $x \in V_M$, $(x)\, A \in \Gamma$ if $A^t/x \in \Gamma$ for all $t \in T_M$.

We will use bold face Greek capitals to range over saturated sets.

By a **Q1**-ω-*extension* of a morphology **M** (abbreviated 'ω-extension' in this section and in Section VI) we understand a morphology **M**′ like **M** except that $C_{M'} = C_M \cup \{c_1, c_2, \ldots\}$, where $c_1, c_2, \ldots$ are symbols foreign to **M**.

L1. Every **Q1**-consistent subset Γ of W_M has an **M**′-saturated extension $\mathbf{\Gamma}$, where **M**′ is any ω-extension of **M**.

The demonstration of this lemma does not differ from that of its classical analogue; all that is needed for its proof are elementary syntactic features of **Q1** and willingness to use the axiom of choice or a like principle. Since every $\mathbf{\Gamma}$ is **Q1**-consistent, we can strengthen L1 a bit.

L2. For any $\Gamma \subseteq W_M$ and ω-extension **M**′ of **M**, Γ is **Q1**-consistent iff Γ has an **M**′-saturated extension $\mathbf{\Gamma}$.

In preparation for our needs in Section VI, below, we establish the following syntactic lemma, which is the crucial step in our proof of the semantic completeness of **Q1**.

L3. Let $\mathbf{\Gamma}$ be any **M**-saturated set, let $\Delta_1 \mathscr{R} \Delta_2$ iff $\{A / \Box A \in \Delta_1\} \subseteq \Delta_2$ and let $\mathscr{K}$ be the closure of $\{\mathbf{\Gamma}\}$ under $\mathscr{R}$. Then $\mathscr{K}$ satisfies the following condition: for all $\Delta \in \mathscr{K}$ and all $A \in W_M$, if $\Diamond A \in \Delta$ then there is a $\Delta' \in \mathscr{K}$ such that $A \in \Delta'$ and $\Delta \mathscr{R} \Delta'$.

Proof. Let $\Theta = \{B / \Box B \in \Delta\}$. Now, the set Θ satisfies condition (3) of D1, since if $B^t/x \in \Theta$ for all $t \in T_M$, then $\Box B^t/x \in \Delta$ for all $t \in T_M$ and hence $(x) \Box B \in \Delta$. But then, by A7, $\Box (x)\, B \in \Delta$, so that $(x)\, B \in \Theta$. Knowing that Θ satisfies condition (3) of D1, it is easy to see that Ξ does, where Ξ is the closure of $\Xi \cup \{A\}$ under $\vdash_1$.

Also Ξ is **Q1**-consistent, since if it were the case that $\Theta \vdash_1 \sim A$, then by T1 we would have $\{\Box B / \Box B \in \Delta\} \vdash_1 \Box \sim A$ and since by assumption $\Diamond A \in \Delta$, Δ would be **Q1**-inconsistent.

The proof which Henkin gives in [3], pp. 3–4, of Theorem 3 of that paper establishes for the classical predicate calculus (without identity) that every consistent subset of W_M satisfying condition (3) of D1 has an **M**-saturated extension, and no changes whatsoever are needed to make this argument work also for **Q1**.[4] Applying this result to Ξ we obtain the desired **M**-saturated extension $\mathbf{\Gamma}$ of $\Theta \cup \{A\}$.

V. SEMANTICS OF Q1[5]

A **Q1**-S4-*model structure* (in the present paper, abbreviated '**Q1**ms') is a triple $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$, where $\mathscr{K}$ is a nonempty set, $\mathscr{R}$ a binary reflexive and transitive relation on $\mathscr{D}$, and $\mathscr{D}$ a non-empty[6] domain.

A **Q1**-*interpretation* I of a morphology **M** on a **Q1**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ is a function which assigns:

(1) To each $x \in V_M$ a member $I(x)$ of $\mathscr{D}$;
(2) To each $c \in C_M$ a member $I(c)$ of $\mathscr{D}$;
(3) To each $P^0 \in P^0_M$ a value $I(P^0)$ in $\{T, F\}$, and to each $P^i \in P^i_M$ $(i > 0)$ a subset $I_\alpha(P^i)$, for each $\alpha \in \mathscr{K}$, of the cartesian product $\mathscr{D}^i$.

Where $d \in \mathscr{D}$, I^d/x is to be the interpretation differing (if at all) from I only in assigning d to x.

The truth-value $I_\alpha(A)$ of A in α under a **Q1**-interpretation I on a **Q1**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ (where $\alpha \in \mathscr{K}$), is defined inductively as follows.

(1) $I_\alpha(Pt_1 \dots t_n) = T$ if $\langle I(t_1), \dots, I(t_n) \rangle \in I_\alpha(P)$,
$I_\alpha(Pt_1 \dots t_n) = F$ otherwise;
(2) $I_\alpha(s = t) = T$ if $I(s) = I(t)$,
$I_\alpha(s = t) = F$ otherwise;
(3) $I_\alpha(A \supset B) = T$ if $I_\alpha(A) = F$ or $I_\alpha(B) = T$,
$I_\alpha(A \supset B) = F$ otherwise;
(4) $I_\alpha(\sim A) = T$ if $I_\alpha(A) = F$,
$I_\alpha(\sim A) = F$ otherwise;
(5) $I_\alpha(\Box A) = T$ if for all $\beta \in \mathscr{K}$ such that $\alpha \mathscr{R} \beta$, $I_\beta(A) = T$,
$I_\alpha(\Box A) = F$ otherwise;
(6) $I_\alpha((x)A) = T$ if for all $d \in \mathscr{D}$, $I^d/x_\alpha(A) = T$,
$I_\alpha((x)A) = F$ otherwise.

The following lemma concerning the relationship of syntactic and semantic substitution is readily proved by induction on the complexity of A.

L4. $I_\alpha(At/x) = I^{I(t)}/x_\alpha(A)$.

An interpretation I on a **Q1**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ *simultaneously satisfies* Γ in α if $I_\alpha(A) = T$ for all $A \in \Gamma$. Where $\Gamma \subseteq W_M$, Γ is *simultaneously* **Q1**-*satisfiable* if there is some **Q1**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$, interpretation I of **M** on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ and $\alpha \in \mathscr{K}$ such that I simultaneously satisfies Γ in α. A formula A is **Q1**-valid if $\{\sim A\}$ is not simultaneously **Q1**-satisfiable.

L5. Let I be an interpretation of **M** on a **Q1**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ such that for all $d \in \mathscr{D}$ there is a $t \in T_M$ such that $I(t) = d$, and let $\alpha \in \mathscr{K}$. Then the set $\Gamma = \{A / I_\alpha(A) = T$ and $A \in W_M\}$ of formulas of **M** simultaneously **Q1**-satisfied by I in α is **M**-saturated.

PROOF. Conditions (2) and (3) of D1 are met trivially. To establish condition (1), one need only verify that A1–A8 are **Q1**-valid and that R1–R3 preserve **Q1**-validity.

VI. SEMANTIC COMPLETENESS OF Q1

L6. Let Γ be **M**-saturated. Then there is an interpretation I of **M** on a **Q1**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ and an $\alpha \in \mathscr{K}$ such that Γ is the set of formulas of **M** simultaneously **Q1**-satisfied by I in α.

PROOF. Let $\mathscr{R}$ be as in L3 and let $\mathscr{K}$ be the closure of $\{\Gamma\}$ under $\mathscr{R}$: i.e. $\mathscr{K}$ is the smallest set S such that $\Gamma \in S$ and for all Δ and Θ, if $\Delta \in S$ and $\Delta \mathscr{R} \Theta$ then $\Theta \in S$. The relation $\simeq$ on T_M such that $s \simeq t$ iff $s = t \in \Gamma$ is an equivalence relation and hence divides T_M into disjoint partitions; let $\mathscr{D}$ be a set of representatives, one from each of these partitions, and let $f(t)$ be the representative of the partition to which t belongs.

The triple $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ is a **Q1**ms; to verify this, we need only check that $\mathscr{R}$ is reflexive and transitive, and this follows at once from A2 and A3.[7]

We now define an interpretation I of **M** on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$, as follows: $I(t) = f(t)$; $I_\Delta(P^0) = T$ if $P^0 \in \Delta$ and $I_\Delta(P^0) = F$ if $P^0 \notin \Delta$; and where $i > 0$, $I_\Delta(P^i) = \{\langle f(t_1), \dots, f(t_i) \rangle / P^i t_1 \dots t_i \in \Delta\}$.

By induction on the complexity of A, we show that for all $A \in$ **M** and $\Delta \in \mathscr{K}$, $I_\Delta(A) = T$ iff $A \in \Delta$. The more interesting cases are the following.

CASE 1. A is $P^i t_1 \dots t_i$. Because of T2 and A8, we have $s \simeq t$ iff $s = t \in \Delta$ for all $\Delta \in \mathscr{K}$. Hence the formulas $t_1 = f(t_1), \dots, t_i = f(t_i)$ are all in each $\Delta \in \mathscr{K}$. Therefore by A6, for all $\Delta \in \mathscr{K}$ we have $A \in \Delta$ iff $P^i f(t_1) \dots f(t_i) \in \Delta$. And by definition of I, this iff $I_\Delta(A) = T$.

CASE 2. A is $s = t$. Now, $I_\Delta(A) = T$ iff $I(s) = I(t)$. But by definition of I, $I_\Delta(s) = I_\Delta(t)$ iff $s = t \in \Gamma$, and, as we know, this iff $s = t \in \Delta$.

CASE 5. A is $\Box B$. Now if $\Box A \in \Delta$, then by definition of $\mathscr{R}$, $B \in \Delta'$ for all Δ' such that $\Delta \mathscr{R} \Delta'$; hence by the hypothesis of induction, $I_{\Delta'}(B) = T$

for all such Δ', and so $I_\Delta(A)=T$. Conversely, if $A\notin\Delta$, then $\Diamond\sim B\in\Delta$ and by L3 there is a $\Delta'\in\mathscr{K}$ such that $\Delta\mathscr{R}\Delta'$ and $\sim B\in\Delta'$; hence $B\notin\Delta'$. By the hypothesis of induction, $I_{\Delta'}(B)=F$, and so $I_\Delta(A)=F$.

CASE 6. A is $(x)B$. By condition (3) of D1, $A\in\Delta$ iff for all $t\in T_M$, $I_\Delta(B^t/x)=T$. Applying L4, we see this is equivalent to the condition that for all $t\in T_M$, $I^{I(t)}/x_\Delta(B)=T$ (i.e. $I^{f(t)}/x_\Delta(B)=T$). And, since $\mathscr{D}=\{f(t)/t\in T_M\}$, this holds iff $I_\Delta(A)=T$.

We have now shown that for all $A\in W_M$ and $\Delta\in\mathscr{K}$, $I_\Delta(A)=T$ iff $A\in\Delta$; hence, in particular, $\Gamma=\{A/I_\Gamma(A)=T$ and $A\in W_M\}$, which was to be proved.

L7. A set Γ is **M**-saturated iff there exists a **Q1**ms $\langle\mathscr{K},\mathscr{R},\mathscr{D}\rangle$, an interpretation I of **M** on $\langle\mathscr{K},\mathscr{R},\mathscr{D}\rangle$ and an $\alpha\in\mathscr{K}$ such that for all $d\in\mathscr{D}$ there is a $t\in T_M$ such that $I(t)=d$, and $\Gamma=\{A/I_\alpha(A)=T\}$.

PROOF. Since the interpretation I defined in the proof of L6 is such that for all $d\in\mathscr{D}$ there is a $t\in T_M$ such that $I(t)=d$, L5 and the proof of L6 together yield the desired result.

L8. Let **M**$'$ be an ω-extension of **M**, and Γ a subset of **M**. Then Γ is simultaneously **Q1**-satisfiable iff Γ has an **M**$'$-saturated extension $\boldsymbol{\Gamma}$.

PROOF. Suppose first that Γ has an **M**$'$-saturated extension $\boldsymbol{\Gamma}$; by L7, $\boldsymbol{\Gamma}$ is simultaneously **Q1**-satisfiable, and hence, so is Γ. Conversely, if Γ is simultaneously **Q1**-satisfiable then by L5, Γ is a subset of a **Q1**-consistent set and hence itself is **Q1**-consistent. Then by L1, Γ has an **M**-saturated extension $\boldsymbol{\Gamma}$.

T1. (Strong semantic completeness of **Q1**). A subset Γ of W_M is **Q1**-consistent iff Γ is **Q1**-satisfiable.

PROOF. Let **M**$'$ be an ω-extension of **M**. By L2, a subset Γ of W_M is **Q1**-consistent iff Γ has an **M**$'$-saturated extension $\boldsymbol{\Gamma}$. But in view of L8, Γ has an **M**$'$-saturated extension $\boldsymbol{\Gamma}$ iff Γ is simultaneously **Q1**-satisfiable.

As usual, T1 yields as corollary the weak semantic completeness of **Q1**.

T2. For all formulas A of **M**, A is **Q1**-valid iff $\vdash_1 A$.

This concludes our treatment of the system **Q1**; we proceed now to an account of **Q3**.

VII. DEDUCIBILITY IN THE SYSTEM Q3

In the system **Q3**, definite descriptions are primitive; therefore, whenever we use the term 'morphology' below, we understand 'morphology with

identity and descriptions'. Any tautology is an axiom of **Q3**, as well as any instance of the following twelve schemes.

A1.	$\Box(A\supset B)\supset.\Box A\supset\Box B$	
A2.	$\Box A\supset A$	
A3.	$\Box A\supset\Box\Box A$	
A4$'$.	$(x)A\supset.\mathrm{E}\Box t\supset A^t/x$,	where t is any term
A5$'$.	$(x)(\mathrm{E}x\supset A)\supset(x)A$	
A6$'$.	$(\exists x)\,\mathrm{E}x$	
A7$'$.	$s=s$	
A8$'$.	$s=t\supset.A\supset A^s//t$,	where no occurrence of t in A that is replaced by s falls within the scope of a modal operator
A9$'$.	$\mathrm{E}\,\imath_x A\supset(\exists!x)A$	
A10$'$.	$(y)((x)(A\equiv x=y)\supset y=\imath_x A)$	
A11$'$.	$x=y\supset\Box x=y$,	where x and y are individual variables
A12$'$.	$\Diamond x=y\supset x=y$,	where x and y are individual variables.

The rules of proof of **Q3** are as follows.[8]

R1. $\dfrac{A \qquad A\supset B}{B}$

R2. $\dfrac{A}{\Box A}$

R3. $\dfrac{A\supset B}{A\supset(x)B}$, where x is not free in A

R4. $\dfrac{A\supset\Box B}{A\supset\Box(x)B}$, where x is not free in A

R5. $\dfrac{A\supset.B_1\prec.\cdots\prec.B_n\prec\Box C}{A\supset.B_1\prec.\cdots\prec.B_n\prec\Box(x)C}$, where x is not free in A, $B_1,\ldots,$ or B_n

R6. $\dfrac{A\supset\sim t=x}{\sim A}$ where x is not free in A or in t

R7. $\dfrac{A\supset.B_1\prec.\cdots\prec.B_n\prec\sim t=x}{A\supset.B_1\prec.\cdots\prec.\Box\sim B_n}$, where x is not free in A, $B_1,\ldots,B_n$, or t.

The definitions of **Q3**-derivability and **Q3**-consistency are carried out as are the analogous definitions in the case of **Q1**. We record here the following facts about **Q3**-derivability.

T3. If $\Gamma \cup \{A^y/x\} \vdash_3 B$, and y is an individual variable not occurring in B or in any member of Γ, then $\Gamma \cup \{(\exists x)A\} \vdash_3 B$.

T4. $\vdash_3 (\exists x)(((\exists x)A \supset A) \wedge \mathrm{E}x)$.

T5. If Γ is **Q3**-consistent and $\Diamond(A_1 \wedge \cdots \wedge A_n) \in \Gamma$, then $\{A_1, \ldots, A_n\}$ is **Q3**-consistent.

T6. $\vdash_3 \mathrm{E}x \equiv \mathrm{E}\Box x$, where x is an individual variable.

T7. $\vdash_3 x = y \supset . A \supset A^x//y$, where x and y are individual variables.

VIII. Q3-SATURATION

The notion of **Q3**-saturation is more complicated than that of **Q1**-saturation. The reason for this is that, in the absence of A7, we must resort to a much more detailed argument to ensure that an analogue of L3 can be proved. First, we define by induction sequences $\mathrm{f}_0, \mathrm{f}_1, \ldots$ and $\mathrm{h}_0, \mathrm{h}_1, \ldots$ of functions. The functions f_n and h_n will be used to guarantee, roughly speaking, that if $\Diamond \cdots \Diamond A \in \Gamma$ (here the $\Diamond$ is repeated n times), then there are saturated sets $\Delta_1, \ldots, \Delta_n$ such that $\Gamma \mathscr{R} \Delta_1$, $\Delta_1 \mathscr{R} \Delta_2, \ldots$, and $\Delta_{n-1} \mathscr{R} \Delta_n$.

D2. $\mathrm{f}_0((\exists x)A, y) = \Diamond(\exists x)A \supset \Diamond(\mathrm{E}y \wedge A^y/x)$

$\mathrm{f}_1(B, (\exists x)A, y) = \Diamond B \supset \Diamond(B \wedge (\Diamond(\exists x)A \supset \Diamond(\mathrm{E}y \wedge A^y/x)))$

$\mathrm{f}_{i+1}(B_1, \ldots, B_{i+1}, (\exists x)A, y)$

$= \Diamond B_1 \supset \Diamond(B_1 \wedge \mathrm{f}_i(B_2, \ldots, B_{i+1}, (\exists x)A, y))$

D3. $\mathrm{h}_1(B, x, t) = \Diamond B \supset \Diamond(B \wedge x = t)$

$\mathrm{h}_{i+1}(B_1, \ldots, B_{i+1}, x, t) = \Diamond B_1 \supset \Diamond(B_1 \wedge \mathrm{h}_i(B_2, \ldots, B_{i+1}, x, t))$

D4. A subset Γ of $\mathrm{W_M}$ is **Q3**-M-*saturated* (abbreviated 'M-saturated' below) if it meets the following seven conditions:[9]

(1) Γ is **Q3**-consistent;

(2) For all $A \in \mathrm{W_M}$, $A \in \Gamma$ or $\sim A \in \Gamma$;

(3) For all $A \in \mathrm{W_M}$ and $x \in \mathrm{V_M}$, $(x)A \in \Gamma$ if $A^y/x \in \Gamma$ for all $y \in \mathrm{V_M}$;

(4) For all $t \in T_{\mathrm{M}}$ there is an $x \in \mathrm{V_M}$ such that $x = t \in \Gamma$;

(5) For all $A \in \mathrm{W_M}$ and $x \in \mathrm{V_M}$, there is a $y \in \mathrm{V_M}$ such that $\mathrm{f}_0((\exists x)A, y) \in \Gamma$;

(6) For all $n > 0$, for all $t \in \mathrm{T_M}$ and $\{B_1, \ldots, B_n\} \subseteq \mathrm{W_M}$, there is an $x \in \mathrm{V_M}$ such that $\mathrm{h}_n(B_1, \cdots, B_n, x, t) \in \Gamma$;

(7) For all $n > 0$, for all $\{B_1, \ldots, B_n, (\exists x)A\} \subseteq \mathrm{W_M}$, there is a $y \in \mathrm{V_M}$ such that $\mathrm{f}_n(B_1, \ldots, B_n, (\exists x)A, y) \in \Gamma$.

Armed with this definition, we proceed in much the same way as before in proving the semantic completeness of the system. Since, however, many adjustments must be made at various points in the classical argument of Henkin [3], we will furnish more details this time.

L9. For all $n > 0$, if $\Gamma \vdash_3 \sim \mathrm{f}_n(B_1, \ldots, B_n, (\exists x)A, y)$ and y does not occur free in $B_1, \ldots, B_n$, $(\exists x)A$, or any member of Γ, then Γ is **Q3**-inconsistent.

PROOF. Induce on n, showing that for all k, if $\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec \sim \mathrm{f}_n(B_1, \ldots, B_n, (\exists x)A, y)$ and y does not occur free in $B_1, \ldots, B_n$, $(\exists x)A$, $C_1, \ldots, C_k$, or any member of Γ, then $\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_{k-1} \prec \Box \sim C_k$. If $\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec \sim \mathrm{f}_1(B, (\exists x)A, y)$, then

(i) $\quad \Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec \Diamond B,$

and

(ii) $\quad \Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec . B \prec (\Diamond(\exists x)A \wedge \Box(\mathrm{E}y \supset \sim A^y/x)).$

From (ii), we see that

(iii) $\quad \Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec . B \prec \Diamond(\exists x)A,$

and

(iv) $\quad \Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec . B \prec \Box(\mathrm{E}y \supset \sim A^y/x).$

Applying R5 to (iv), we have

$$\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec . B \prec \Box(x)(\mathrm{E}x \supset \sim A);$$

hence, by A5′,

(v) $\quad \Gamma \vdash_3 C_1 \prec . \cdots \prec C_k \prec . B \prec \Box(x) \sim A.$

But (iii) and (v) yield $\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec \Box \sim B$, which together with (i) yields what was to be shown, that

$$\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_{k-1} \prec \Box \sim C_k$$

(or, in case $k = 0$, that Γ is **Q3**-inconsistent).

Suppose now that the property to be established holds for $n = j$. If $\Gamma \vdash_3 C_1 \prec . \cdots \prec . \sim \mathrm{f}_{j+1}(B_1, \ldots, B_n, (\exists x)A, y)$ then

(vi) $\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec \Diamond B_1$

and

(vii) $\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec . B_1 \prec \sim f_j(B_1, \ldots, B_j, (\exists x)A, y)$.

Applying the hypothesis of induction to (vii), we have

$$\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_k \prec \Box \sim B_1,$$

which together with (vi) yields

$$\Gamma \vdash_3 C_1 \prec . \cdots \prec . C_{k-1} \prec \Box \sim C_k.$$

The lemma, being a special case of the result we have just proved inductively, is now established.

L10. For all $n>0$, if $\Gamma \vdash_3 \sim h_n(B_1, \ldots, B_n, x, t)$ and x does not occur free in $B_1, \ldots, B_n, t$, or any member of Γ, then Γ is **Q3**-inconsistent.

Proof: like the proof of L9, but making use of R7 instead of R5.

By a **Q3**-*ω-extension* (abbreviated '*ω*-extension' below) of **M**, we understand a morphology **M′** like **M** except that $V_{M'} = V_M \cup \{z_1, z_2, \ldots\}$, where $z_1, z_2, \ldots$ are symbols foreign to **M**.

L11. Every **Q3**-consistent subset Γ of W_M has an **M′**-saturated extension Γ, where **M′** is any *ω*-extension of **M**.

Proof. Let $Z = \{z_1, z_2, \ldots\}$ be the set of individual variables added to **M** in passing to **M′**. In saturating Γ, we will use a limiting construction in which denumerably many things are done denumerably many times; to index these operations we take a partitioning of the nonnegative integers into denumerably many denumerable sets. $S_0, S_1, S_2, \ldots$. Let Γ_0 be Γ, and define Γ_{i+1} inductively according to the following cases.

(1) $i \in S_0$. Let $(\exists x)A$ be the alphabetically first formula of **M′** of the kind $(\exists y)B$ such that for all $z \in Z$, $((\exists x)A \supset A^z/x) \wedge Ez) \notin \Gamma_i$, and let z be the first member of Z not to occur in any member of Γ_i or in $(\exists x)A$. Then let $\Gamma_{i+1} = \Gamma_i \cup \{((\exists x)A \supset A^z/x) \wedge Ez)\}$.

(2) $i \in S_1$. Let $(\exists x)A$ be the alphabetically first formula of **M** of the kind $(\exists y)B$ such that for all $z \in Z$, $f_0((\exists x)A, z) \notin \Gamma_i$, and let z be the first member of Z not to occur in any member of Γ_i or in $(\exists x)A$. Then let $\Gamma_{i+1} = \Gamma_i \cup \{f_0((\exists x)A, z)\}$.

(3) $i \in S_2$. Let t be the alphabetically first term of **M′** such that for all $z \in Z$, $z = t \notin \Gamma_i$, and let z be the alphabetically first member of Z not to occur in any member of Γ_i or in t. Then let $\Gamma_{i+1} = \Gamma_i \cup \{z = t\}$.

(4) $i \in S_{2n+1}$, where $n>0$. Let $B_1 \vee \cdots \vee B_n \vee (\exists x)A$ be the alphabetically first formula of **M′** of the kind $C_1 \vee \cdots \vee C_n \vee (\exists y)D$ such that for all $z \in Z$, $f_n(B_1, \ldots, B_n, (\exists x)A, z) \notin \Gamma_i$, and let z be the first member of Z not to occur in any member of Γ_i or in $B_1 \vee \cdots \vee B_n \vee (\exists x)A$. Then let $\Gamma_{i+1} = \Gamma_i \cup \{f_n(B_1, \ldots, B_n, (\exists x)A, z)\}$.

(5) $i \in S_{2n+2}$, where $n>0$. Let $B_1 \vee \cdots \vee B_n \vee t = t$ be the alphabetically first formula of **M′** of the kind $C_1 \vee \cdots \vee C_n \vee s = s$ such that for all $z \in Z$, $h_n(B_1, \ldots, B_n, z, t) \in \Gamma_i$, and let z be the first member of Z not to occur in any member of Γ_i or in $B_1 \vee \cdots \vee B_n \vee t = t$. Then let $\Gamma_{i+1} = \Gamma_i \cup \{h_n(B_1, \ldots, B_n, z, t)\}$.

We now show by induction that for all i, Γ_i is **Q3**-consistent. In Case 1, if Γ_i were **Q3**-inconsistent we would have

$$\Gamma_i \cup \{((\exists x)A \supset A^z/x) \wedge Ez\} \vdash_3 P \wedge \sim P,$$

and hence by T3

$$\Gamma_i \cup \{(\exists x)(((\exists x)A \supset A) \wedge Ex)\} \vdash_3 P \wedge \sim P,$$

But then, by T4, we would have

$$\Gamma_i \vdash_3 P \wedge \sim P$$

and Γ_i would be **Q3**-inconsistent.

In Case 2, if Γ_{i+1} were **Q3**-inconsistent we would have

(i) $\Gamma_i \vdash_3 \Diamond (\exists x)A$

and

(ii) $\Gamma_i \vdash_3 \Box (Ez \supset \sim A^z/x)$.

But then, applying R4 to (ii), we would have

$$\Gamma_i \vdash_3 \Box (x)(Ex \supset \sim A),$$

and hence by A5′,

$$\Gamma_i \vdash_3 \Box [(x)] \sim A.$$

Putting this together with (i), we see that Γ_i would be **Q3**-inconsistent.

In Case 3, if Γ_{i+1} were **Q3**-inconsistent, we would have $\Gamma_i \vdash_3 \sim z = t$ and so, because of R6, Γ_i would be **Q3**-inconsistent.

In Case 4, if Γ_{i+1} were **Q3**-inconsistent, where $i \in S_{2n+1}$ and $n>0$, we would have

$$\Gamma_i \vdash_3 \sim \mathrm{f}_n(B_1, \ldots, B_n, (\exists x)A, z).$$

Applying L9, we see that under these circumstances Γ_i would be **Q3**-inconsistent.

In Case 5, if Γ_{i+1} were **Q3**-inconsistent where $i \in \mathrm{S}_{2n+2}$ and $n>0$, we would have

$$\Gamma_i \vdash_3 \sim \mathrm{h}_n(B_1, \ldots, B_n, z, t).$$

Applying L10, we see that under these circumstances Γ_i would be **Q3**-inconsistent.

We have now shown that for all i, Γ_i is **Q3**-consistent; therefore $\Delta = \bigcup_{i \in \omega} \Gamma_i$ is **Q3**-consistent. Extending Δ in the usual way to a negation-complete set, we obtain the desired saturated extension of Γ.

Again, we can strengthen L11.

L12. For any $\Gamma \subseteq \mathrm{W_M}$ and ω-extension M′ of M, Γ is **Q3**-consistent iff Γ has an M′-saturated extension Γ.

Having established analogues for **Q3** of L1 and L2, we turn to the problem of doing the same for L3. With the completion of L13, below, the most difficult part of the completeness proof will be finished.

L13. Let Γ be any M-saturated set, let $\Delta_1 \mathscr{R} \Delta_2$ iff $\{A/\Box A \in \Delta_1\} \subseteq \Delta_2$, and let $\mathscr{K}$ be the closure of $\{\Gamma\}$ under $\mathscr{R}$. Then $\mathscr{K}$ satisfies the following condition: for all $\Delta \in \mathscr{K}$ and all $A \in \mathrm{W_M}$, if $\Diamond A \in \Delta$ then there is a $\Delta' \in \mathscr{K}$ such that $A \in \Delta'$ and $\Delta \mathscr{R} \Delta'$.

PROOF.[10] Suppose that $\Delta \in \mathscr{K}$ and that $\Diamond A \in \Delta$. Define by induction a sequence $B_0, B_1, \ldots$ of formulas of M, as follows. Let A be B_0, and as in the proof of L11, above, let $\mathrm{S}_0, \mathrm{S}_1, \ldots$ be a partitioning of the nonnegative integers into denumerably many denumerable sets. Define B_{i+1} by cases in the following way.

(1) If $i \in \mathrm{S}_0$ let C be the alphabetically first formula of M such that neither $C \in \{B_0, \ldots, B_i\}$ nor $\sim C \in \{B_0, \ldots, B_i\}$. Then let B_{i+1} be C if $\Diamond(B_0 \wedge \cdots \wedge B_i \wedge C) \in \Delta$, and $\sim C$ otherwise,

(2) If $i \in \mathrm{S}_1$ then in case there is no formula of the kind $(\exists y)D \in \{B_0, \ldots, B_i\}$, let B_{i+1} be B_i. And in case there is such a formula, let $(\exists x)C$ be the alphabetically first formula, and let z be the first member of $\mathrm{V_M}$ such that

$$\Diamond(B_0 \wedge \cdots \wedge B_i) \supset \Diamond(B_0 \wedge \cdots \wedge B_i \wedge \mathrm{E}z \wedge C^z/x) \in \Delta.$$

Then let B_{i+1} be $\mathrm{E}z \wedge C^z/x$.

(3) If $i \in \mathrm{S}_2$, then let t be the alphabetically first term of M such that $\Diamond(B_0 \wedge \cdots \wedge B_i) \supset \Diamond(B_0 \wedge \cdots \wedge B_i \wedge z = t) \in \Delta$. Then let B_{i+1} be $z = t$.

(4) If $i \in \mathrm{S}_3$, then in case there is no formula of the kind $\Diamond(\exists y)D \in \{B_0, \ldots, B_i\}$, let B_{i+1} be B_i. And in case there is such a formula, let $(\exists x)C$ be the alphabetically first such formula, and let z be the first member of $\mathrm{V_M}$ such that

$$\Diamond(B_0 \wedge \cdots \wedge B_i) \supset \Diamond(B_0 \wedge \cdots \wedge B_i \wedge \mathrm{f}_0((\exists x)C, z)) \in \Delta.$$

Then let B_{i+1} be $\mathrm{E}z \wedge C^z/x$.

(5) If $i \in \mathrm{S}_{2n+2}$ (where $n>0$), let $C_1 \vee \cdots \vee C_n \vee t = t$ be the alphabetically first formula of M of the kind $D_1 \vee \cdots \vee D_n \vee s = s$ such that for all $x \in \mathrm{V_M}$, $\mathrm{h}_n(C_1, \ldots, C_n, x, t) \notin \{B_0, \ldots, B_i\}$, and let z be the alphabetically first member of $\mathrm{V_M}$ such that

$$\Diamond(B_1 \wedge \cdots \wedge B_i) \supset \Diamond(B_1 \wedge \cdots \wedge B_i \wedge \mathrm{h}_n(C_1, \ldots, C_n, z, t)) \in \Delta.$$

Then let B_{i+1} be $z = t$.

(6) If $i \in \mathrm{S}_{2n+3}$ (where $n>0$), let $C_1 \vee \cdots \vee C_n \vee (\exists x)C$ be the alphabetically first formula of the kind $D_1 \vee \cdots \vee D_n \vee (\exists y)D$ such that for all $x \in \mathrm{V_M}$, $\mathrm{f}_n(C_1, \ldots, C_n, (\exists x)C, x) \notin \{B_0, \ldots, B_i\}$, and let z be the first member of $\mathrm{V_M}$ such that

$$\Diamond(B_0 \wedge \cdots \wedge B_i) \supset \Diamond(B_0 \wedge \cdots \wedge B_i \wedge \mathrm{f}_n(C_1, \ldots, C_n, (\exists x)C, z)) \in \Delta.$$

Then let B_{i+1} be $\mathrm{f}_n(C_1, \ldots, C_n, (\exists x)C, z)$.

Now we claim that for all i, B_i is defined and $\Diamond(B_0 \wedge \cdots \wedge B_i) \in \Delta$. This is easily shown by induction; our assumption that $\Diamond A \in \Delta$ furnishes the basis case. In Case 1 of the construction, it is clear that B_{i+1} is defined, and $\Diamond(B_0 \wedge \cdots \wedge B_{i+1}) \in \Delta$ since for all formulas C, if it were the case that $\Diamond(B_0 \wedge \cdots \wedge B_i \wedge C) \notin \Delta$ and $\Diamond(B_0 \wedge \cdots \wedge B_i \wedge \sim C) \notin \Delta$ then $(B_0 \wedge \cdots \wedge B_i) \prec C \in \Delta$ and $(B_0 \wedge \cdots \wedge B_i) \prec \sim C \in \Delta$ so that we would have $\Diamond(B_0 \wedge \cdots \wedge B_i) \notin \Delta$, contrary to assumption.

In Case 2, let u be the alphabetically first variable of M not to occur in any member of $\{B_0, \ldots, B_i\}$, and let $(\exists x)C$ be B_k. Then since $\Diamond(B_0 \wedge \cdots \wedge B_i) \in \Delta$, $\Diamond(\exists u)\ (B_0 \wedge \cdots \wedge B_{k-1} \wedge C^u/x \wedge B_{k+1} \wedge \cdots \wedge B_i) \in \Delta$. The M-saturation of Δ guarantees that $\Diamond(\exists u)\ (B_0 \wedge \cdots \wedge B_{k-1} \wedge C^u/x \wedge B_{k+1} \wedge \cdots \wedge B_i) \supset \Diamond(\mathrm{E}z \wedge B_0 \wedge \cdots \wedge B_{k-1} \wedge C^z/x \wedge B_{k+1} \wedge \cdots \wedge B_i) \in \Delta$ for some $z \in V_M$; and hence, since $\vdash_3 (Ez \wedge C^z/x) \supset (\exists x)C$, we have $\diamond(B_0 \wedge \cdots \wedge B_i \wedge Ez \wedge C^z/x) \in \Delta$; i.e. $\diamond(B_0 \wedge \cdots \wedge B_{i+1}) \in \Delta$.

In Case 3, B_{i+1} is again defined because the **M**-saturation of Δ guarantees that there is a $z \in V_M$ such that $h_1(B_0 \wedge \cdots \wedge B_i, z, t) \in \Delta$; in Case 4, B_{i+1} is defined because for some $z \in V_M$, $f_i(B_0, \ldots, B_i, (\exists x)C, z) \in \Delta$. For the same reason, B_{i+1} is defined in Cases 4 and 5, and again the instantiating variable is chosen so that $\diamond(B_0 \wedge \cdots \wedge B_{i+1}) \in \Delta$ if $\diamond(B_0 \wedge \cdots \wedge B_i) \in \Delta$.

Now let $\Delta' = \bigcup_{i \in \omega} B_i$. Clearly, $A \in \Delta'$; we further claim that Δ' is **M**-saturated and that $\Delta \mathscr{R} \Delta'$. The set Δ' is **Q3**-consistent since for every finite subset $\{C_1, \ldots, C_n\}$ of Δ', $\diamond(C_1 \wedge \cdots \wedge C_n) \in \Delta$ and hence by T5, $\{C_1, \ldots, C_n\}$ is **Q3**-consistent. And Δ' is negation-complete on **M** since Case 1 of the construction ensures that for all C, either $C \in \Delta'$ or $\sim C \in \Delta'$. Similarly, Cases 2, 3, 4, 5, and 6 of the construction guarantee that Δ' fulfills clauses 3, 4, 5, 6, and 7, respectively, of D4. Thus, Δ' is **M**-saturated.

Furthermore, suppose that $\Box C \in \Delta$; then $C \in \Delta'$, since if $\sim C \in \Delta'$ then we would have $\diamond \sim C \in \Delta$, contrary to assumption. Therefore $\Delta \mathscr{R} \Delta'$.

This completes the proof of L13.

IX. SEMANTICS OF Q3

A **Q3-S4**-*model structure* (in the present paper, abbreviated '**Q3**ms') is a quadruple $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$, where $\mathscr{K}$ is a nonempty set, $\mathscr{R}$ a binary reflexive and transitive relation on $\mathscr{K}$, $\mathscr{D}$ a function taking members α of $\mathscr{K}$ into nonempty domains $\mathscr{D}_\alpha$, and $\mathscr{D}'$ a set disjoint with $\bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha$, such that for all $\beta \in \mathscr{K}$, $(\mathscr{D}' \cup \bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha) - \mathscr{D}_\beta$ is nonempty.

Let $\mathscr{D} = \mathscr{D}' \cup \bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha$ be the set of all individuals associated with the **Q3**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$. A **Q3**-*interpretation* I of a morphology **M** on a **Q3**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ is a function which, for each $\alpha \in \mathscr{K}$, assigns:

(1) To each $x \in V_M$ a member $I(x)$ of $\mathscr{D}$;
(2) To each $c \in C_M$ a member $I_\alpha(c)$ of $\mathscr{D}$;
(3) To each $P^0 \in P^0_M$ a value $I_\alpha(P^0)$ in $\{T, F\}$ and to each $P^i \in P^i_M (i > 0)$ a subset $I_\alpha(P^i)$ of the cartesian product $\mathscr{D}^i$.

Again, where $d \in \mathscr{D}$, I^d/x is the interpretation differing (if at all) from I only in assigning d to x.

The truth-value $I_\alpha(A)$ in α of a formula A under a **Q3**-interpretation I on a **Q3**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$, and the value $I_\alpha(t)$ assigned in α to a term t under such an interpretation, are defined by simultaneous induction. This time we omit the clauses for sentential connectives, which are exactly like the corresponding clauses in the case of **Q1**. First, let $I_\alpha(x) = I(x)$, for all $\alpha \in \mathscr{K}$.

(1) $I_\alpha(Pt_1 \ldots t_n) = T$ if $\langle I_\alpha(t_1), \ldots, I_\alpha(t_n) \rangle \in I_\alpha(P)$,
$I_\alpha(Pt_1 \ldots t_n) = F$ otherwise;
(2) $I_\alpha(\imath_x A) =$ the unique $d \in \mathscr{D}_\alpha$ such that $I^d/x_\alpha(A) = T$, if there is such an individual d, $I_\alpha(\imath_x A) =$ an arbitrary[11] member of $\mathscr{D}' - \mathscr{D}_\alpha$ otherwise;[12]
(3) $I_\alpha(s = t) = T$ if $I_\alpha(s) = I_\alpha(t)$,
$I_\alpha(s = t) = F$ otherwise;
(7) $I_\alpha((x)A) = T$ if for all $d \in \mathscr{D}_\alpha$, $I^d/x_\alpha(A) = T$,
$I_\alpha((x)A) = F$ otherwise.

Again, we record for later use a lemma concerning substitution.

L14. Let I be an interpretation of **M** on a **Q3**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ and let $I(y) = d$, where $y \in V_M$. Then $I^d/x_\alpha(A) = I_\alpha(A^y/x)$.

The notions of *simultaneous* **Q3**-*satisfiability* and of **Q3**-*validity* are defined as in the case of **Q1**.

L15. Let I be an interpretation of **M** on a **Q3**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ such that for all $d \in \mathscr{D}$ there is an $x \in V_M$ such that $I(x) = d$. Let $\alpha \in \mathscr{K}$. Then the set $\Gamma = \{A / I_\alpha(A) = T \text{ and } A \in W_M\}$ of formulas of **M** simultaneously **Q3**-satisfied by I in α is **M**-saturated.

Proof. Again, we can easily check that the axioms of **Q3** are all **Q3**-valid and that **Q3**-validity is preserved by the rules of proof of **Q3**. Condition (1), as before, follows immediately from this, and condition (2) is trivial. To establish condition (3), suppose that $(x)A \in \Gamma$; then for some $d \in \mathscr{D}_\alpha$, $I^d/x_\alpha(A) = F$. Let $I(y) = d$; by L13, $I_\alpha(A^y/x) = F$, and y has been chosen so that $Ey \in \Gamma$. Conditions (4)–(7) are verified in the same way.

X. SEMANTIC COMPLETENESS OF Q3

L16. Let Γ be **M**-saturated. Then there is an interpretation I of **M** on a **Q3**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ and an $\alpha \in \mathscr{K}$ such that Γ is the set of formulas of **M** simultaneously **Q3**-satisfied by I in α.

Proof. Let $\mathscr{R}$ be as in L13, and let $\mathscr{K}$ be the closure of $\{\Gamma\}$ under $\mathscr{R}$. The relation $\simeq$ on V_M such that $x \simeq y$ iff $x = y \in \Gamma$ is an equivalence relation and hence divides V_M into disjoint partitions; let $\mathscr{D}^*$ be a set of representatives, one from each of these partitions, and let $f(x)$ be the representative of the partition to which x belongs. Define a function $\mathscr{D}$ from $\mathscr{K}$ into subsets of $\mathscr{D}^*$, as follows: $\mathscr{D}_\Delta = \{f(x)/x \in V_M$ and $Ex \in \Delta\}$. Since $\vdash_3 (\exists x)\, Ex$, $\mathscr{D}_\Delta$ is nonempty for all $\Delta \in \mathscr{K}$. Finally, let $\mathscr{D}' = \{f(x)/x \in V_M$ and for all $\Delta \in \mathscr{K}$, $\sim Ex \in \Delta\}$. It is easily verified that $\mathscr{D}' = \mathscr{D}^* - \bigcup_{\Delta \in \mathscr{K}} \mathscr{D}_\Delta$. Also, for all $\Delta \in \mathscr{K}$, $\mathscr{D}^* - \mathscr{D}_\Delta$ is nonempty, for because Δ is M-saturated there is a $y \in V_M$ such that $y = \imath_x(P \wedge \sim P) \in \Delta$; but $f(y) \in \mathscr{D}^*$ and $f(y) \notin \mathscr{D}_\Delta$.

A2 and A3 again ensure that $\mathscr{R}$ is reflexive and transitive; it follows that the quadruple $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ is a **Q3**ms. We now define an interpretation I of **M** on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$.

For all $t \in T_M$ and all $\Delta \in \mathscr{K}$, there is an $x \in V_M$ such that $x = t \in \Delta$; let $g_\Delta(t)$ be $f(x)$. (The value $g_\Delta(t)$ is independent of the choice of x, since if $x = t \in \Delta$ and $y = t \in \Delta$, then $x = y \in \Delta$ and hence $\Diamond x = y \in \Gamma$; therefore $x = y \in \Gamma$ and $f(x) = f(y)$.) Notice that for all $x \in V_M$ and $\Delta \in \mathscr{K}$, $f(x) = g_\Delta(x)$.

For all $x \in V_M$, let $I(x) = f(x)$; for all $c \in C_M$, let $I_\Delta(c) = g_\Delta(c)$; for all $P^0 \in P^0_M$, let $I_\Delta(P^0) = T$ if $P^0 \in \Delta$ and $I_\Delta(P^0) = F$ if $P^0 \notin \Delta$; and where $i > 0$, for all $P^i \in P^i_M$ let $I_\Delta(P^i) = \{\langle f(x_1), \ldots, f(x_i) \rangle / P^i x_1 \ldots x_i \in \Delta\}$. In case there is no unique $d \in \mathscr{D}_\Delta$ such that $I^d/x_\Delta(A) = T$, let $I_\Delta(\imath_x A)$ be $g_\Delta(\imath_x A)$. To ensure that I, thus defined, is an interpretation of **M** on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ we must show that under these circumstances $g_\Delta(\imath_x A) \in \mathscr{D}_\Delta$. This will follow from the argument below.

By simultaneous induction on the complexity of A and t, we show that for all $A \in W_M$, $t \in T_M$, and $\Delta \in \mathscr{K}$, $I_\Delta(A) = T$ iff $A \in \Delta$, and that $I_\Delta(t) = g_\Delta(t)$. We will omit the Cases (6, 7, and 8) of the induction concerning sentential connectives; the remaining cases are as follows.

CASE 1. A is $P^i t_1 \ldots t_i$. By the definition of satisfaction, $I_\Delta(P^i t_1 \ldots t_i) = T$ iff $\langle I_\Delta(t_1), \ldots, I_\Delta(t_i) \rangle \in I_\Delta(P^i)$. By the hypothesis of induction, $I_\Delta(t_k) = g_\Delta(t_k)$, for $1 \leqslant k \leqslant i$; hence, $\langle I_\Delta(t_1), \ldots, I_\Delta(t_i) \rangle \in I_\Delta(P^i)$ iff $\langle g_\Delta(t_1), \ldots, g_\Delta(t_i) \rangle \in I_\Delta(P^i)$. Now, for all k, $1 \leqslant k < i$, there is an $x_k \in V_M$ such that $x_k = t_k \in \Delta$ and $f(x_k) = g_\Delta(t_k)$; therefore $\langle g_\Delta(t_1), \ldots, g_\Delta(t_i) \rangle \in I_\Delta(P^i)$ iff $\langle f(x_1), \ldots, f(x_i) \rangle \in I_\Delta(P^i)$. But by the definition of I, $\langle f(x_1), \ldots, f(x_i) \rangle \in I_\Delta(P^i)$ iff $P^i x_1 \ldots x_n \in \Delta$, and in view of A8′, $P^i x_1 \ldots x_n \in \Delta$ iff $P^i t_1 \ldots t_i \in \Delta$.

CASE 2. The term t is x where $x \in V_M$. By definition, $I_\Delta(t) = f(x) = g_\Delta(x)$.

CASE 3. The term t is c, where $c \in C_M$. By definition, $I_\Delta(c) = g_\Delta(c)$.

CASE 4. A is $s = t$. By the hypothesis of induction, $I_\Delta(t) = g_\Delta(t)$ and $I_\Delta(s) = g_\Delta(s)$. By the definition of satisfaction, $I_\Delta(s = t) = T$ iff $I_\Delta(s) = I_\Delta(t)$, and this holds iff $g_\Delta(s) = g_\Delta(t)$. But this holds iff there is an $x \in V_M$ such that $x = s \in \Delta$ and $x = t \in \Delta$, and (again, using A8′), this holds iff $s = t \in \Delta$.

CASE 5. The term t is $\imath_x B$. Suppose first that there is no unique $d \in \mathscr{D}_\Delta$ such that $I^d/x_\Delta(B) = T$. Then by definition, $I_\Delta(t) = g_\Delta(t)$. On the other hand, suppose that there exists a unique $f(y) \in \mathscr{D}_\Delta$ such that $I^{f(y)}/x_\Delta(B) = T$. Using properties of M-saturation, it is easy to see that in this case, $(x)(B \equiv x = f(y)) \in \Delta$. But then, in view of A10′ and A4′, we have $E\Box f(y) \supset f(y) = \imath_x B \in \Delta$; and by T6, $E\Box f(y) \in \Delta$ since $Ef(y) \in \Delta$, so that $f(y) = \imath_x B \in \Delta$. Therefore, $g_\Delta(\imath_x B) = f(y)$. But, by the definition of satisfaction, $I_\Delta(\imath_x B) = f(y)$.

CASE 9. A is $(x)B$. By condition (3) of D3, $A \in \Delta$ iff for all $y \in V_M$ such that $Ey \in \Delta$, $B^y/x \in \Delta$. By T7, this holds iff for all $y \in \mathscr{D}_\Delta$, $B^y/x \in \Delta$; and by the hypothesis of induction, this in turn holds iff for all $y \in \mathscr{D}_\Delta$, $I_\Delta(B^y/x) = T$. By L14, this is equivalent to the condition that for all $y \in \mathscr{D}_\Delta$, $I^y/x_\Delta(B) = T$; and by the definition of satisfaction, this iff $I_\Delta(B) = T$.

Now that this property of I has been established by induction, we can return to the problem of showing I to be an interpretation. Suppose that there is no unique $d \in \mathscr{D}_\Delta$ such that $I^d/x_\Delta(A) = T$. Then, clearly, $(\exists! x) A \notin \Delta$, and so, by A9′, $E\imath_x A \notin \Delta$. But then by A8′, $E(g_\Delta(\imath_x A)) \notin \Delta$ and hence $I_\Delta(\imath_x A) \notin \mathscr{D}_\Delta$. Thus, I is an interpretation of **M** on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$.

The induction above establishes that $\Gamma = \{A / I_\Gamma(A) = T\}$, and so L16 is proved.

The following lemmas and theorems are proved in the same way as L7, L8, T1, and T2, above.

L17. A set Γ is M-saturated iff for some **Q3**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ and interpretation I of **M** on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ such that for all $d \in \mathscr{D}^*$ there is an $x \in V_M$ such that $I(x) = d$, there exists an $\alpha \in \mathscr{K}$ such that $\Gamma = \{A / I_\alpha(A) = T\}$.

L18. Let **M**′ be an ω-extension of **M**, and Γ a subset of W_M. Then Γ is simultaneously **Q3**-satisfiable iff Γ has an **M**′-saturated extension Γ.

T3. (Strong semantic completeness of **Q3**).[13] A subset Γ of W_M is **Q3**-consistent iff Γ is simultaneously **Q3**-satisfiable.

T4. For all formulas A, A is **Q3**-valid iff $\vdash_3 A$.

With these theorems, the main theme of this paper is completed. We conclude with a brief account of how T3 may be used to demonstrate the semantic completeness of yet another system of modal predicate calculus.

XI. THE SYSTEM $\mathbf{Q3^p}$

This system is a deductive extension of **Q3**; it is obtained by adding an axiom of permanence, $(x)\Box Ex$, to the axioms and rules of **Q3**. Intuitively, the meaning of this axiom is that no individual ever passes out of existence. The system $\mathbf{Q3^p}$ is closely related to the system of modal predicate calculus of Hintikka [5], chapter 6.

A $\mathbf{Q3^p}$ms can be defined by adding to the definition of a Q3ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ the requirement that for all $\alpha, \beta \in \mathscr{K}$, $\mathscr{D}_\alpha \subseteq \mathscr{D}_\beta$. In the usual way, this yields corresponding notions of simultaneous $\mathbf{Q3^p}$-satisfiability and of $\mathbf{Q3^p}$-validity.

T5. (Strong semantic completeness of $\mathbf{Q3^p}$). A subset Γ of W_M is $\mathbf{Q3^p}$-consistent iff Γ is simultaneously $\mathbf{Q3^p}$-satisfiable.

PROOF. Γ is $\mathbf{Q3^p}$-consistent iff $\Gamma \cup \{(x)\Box Ex\}$ is **Q3**-consistent. By T3, $\Gamma \cup \{(x)\Box Ex\}$ is **Q3**-consistent iff $\Gamma \cup \{(x)\Box Ex\}$ is simultaneously **Q3**-satisfiable. Let $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ be a **Q3**ms, and I an interpretation of **M** on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ such that for some $\alpha \in \mathscr{K}$, $I_\alpha((x)\Box Ex) = T$. Let $\mathscr{K}^*$ be the closure of $\{\alpha\}$ under $\mathscr{R}$; clearly, $\langle \mathscr{K}^*, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ is a $\mathbf{Q3^p}$ms, and $I'_\alpha(A) = T$ iff $I_\alpha(A) = T$, where I′ is the restriction of I to $\langle \mathscr{K}^*, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$. Therefore, if $\Gamma \cup \{(x)\Box Ex\}$ is simultaneously $\mathbf{Q3^p}$-satisfiable then Γ is $\mathbf{Q3^p}$-satisfiable. On the other hand, since every $\mathbf{Q3^p}$ms is a Q3ms, if Γ is $\mathbf{Q3^p}$-satisfiable then $\Gamma \cup \{(x)\Box Ex\}$ is **Q3**-satisfiable. Therefore, Γ is $\mathbf{Q3^p}$-consistent iff Γ is simultaneously $\mathbf{Q3^p}$-satisfiable.

As usual, we obtain as a corollary the weak semantic completeness of $\mathbf{Q3^p}$.

T6. For all formulas A of **M**, A is $\mathbf{Q3^p}$-valid iff A is a theorem of $\mathbf{Q3^p}$.

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BIBLIOGRAPHY

[1] Church, A., *Introduction to Mathematical Logic*, vol. I, Princeton 1956.
[2] Cocchiarella, N., *Tense Logic: a Study of Temporal Reference*. Dissertation, The University of California, Los Angeles, 1966.
[3] Henkin, L., 'The Completeness of the First-Order Functional Calculus', *Journal of Symbolic Logic* **14** (1949) 159–166.
[4] Henkin, L., 'A Generalization of the Concept of ω-Completeness', *Journal of Symbolic Logic* **22** (1957) 1–14.

[5] Hintikka, J., *Knowledge and Belief*. Ithaca, New York, 1962.
[6] Kripke, S., 'A Completeness Theorem in Modal Logic', *Journal of Symbolic Logic* **24** (1959) 1–14.
[7] Kripke, S., 'Semantical Analysis of Modal Logic I: Normal Propositional Calculi', *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **9** (1963) 67–96.
[8] Kripke, S., 'Semantical Considerations on Modal and Intuitionistic Logic I' in *Proceedings of a Colloquium on Modal and Many-Valued Logics*, Helsinki 1963, pp. 83–94.
[9] Kripke, S., 'Semantical Analysis of Intuitionistic Logic I' in *Formal Systems and Recursive Functions* (ed. by J. Crossley and M. Dummett), Amsterdam 1965, pp. 92–130.
[10] Montague, R. and L. Henkin, 'On the Definition of "Formal Deduction"', *Journal of Symbolic Logic* **21** (1957) 129–136.
[11] Stalnaker, R. and R. Thomason, 'Abstraction in First-Order Modal Logic', *Theoria* (Lund), **14** (1968) 203–207.
[12] Thomason, R., 'On the Strong Semantical Completeness of the Intuitionistic Predicate Calculus', *Journal of Symbolic Logic* **33** (1968) 1–7.
[13] Thomason, R., 'Modal Logic and Metaphysics' in *The Logical Way of Doing Things* (ed. by K. Lambert), New Haven, Conn., 1969, pp. 119–146.
[14] van Fraassen, B. and K. Lambert, 'On Free Description Theory', *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **13** (1967) 225–240.

REFERENCES

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1 The system Q2 is discussed in Thomason [13], where a definition of Q2-validity is given. Prof. David Kaplan has informed me (in a private communication, April, 1967) that the notion of Q2-validity cannot be recursively axiomatized.

2 The system Q3 is a generalization of the version of modal predicate calculus described in Kripke [8], which treats only closed formulas, and hence gives no account of individual constants or definite descriptions. On the other hand, the system Q1S5 of modal predicate calculus based on an S5-type modality and on a Q1-type theory of quantification and identity is the system proved semantically complete in Kripke [6]. In this case our generalization of Kripke's results consists in allowing for sorts of modality other than S5, and in proving strong rather than weak completeness.

3 We will use dots in the usual way in place of parentheses; see Church [1], pp. 74–80.

4 This proof requires that the morphology (i.e. the set of formulas of the morphology) be denumerable.

5 See the articles of Kripke, especially [7] and [8], for an intuitive account of this semantics. Another discussion of this sort may be found in Thomason [13].

6 The requirement that the domains be nonempty is easily lifted; in this case, one must also drop A6′ from the system Q3.

7 This is the only place in the proof of semantic completeness which must be changed to adjust the argument to kinds of modality other than S4.

8 The rules R4–R7 are needed for the proof of semantic completeness of Q3. At present, it is not known whether these rules are redundant.