Karel Lambert, ed., *The Logical Way of Doing Things*, Yale University Press, New Haven, 1969.

6. Modal Logic and Metaphysics[1]

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1. In recent years, the research of many logicians has converged toward a solution of the problem of quantification into modal contexts, and in the last decade a new level of understanding has been achieved in this area.[2] Of course, problems still remain; but these can now be treated within the context of a well-articulated logical theory. In this paper I wish to discuss in an informal way some of my own contributions to this development; these consist primarily of semantical completeness theorems for some systems of modal logic with quantifiers.

Perhaps the simplest example of a theorem of this sort is the one obtained by Post in the 1920s for the familiar sentential calculus. Post showed that every formula provable in this system was tautologous, and conversely that every tautology was a theorem: thereby establishing that a notion of *validity* for such formulas coincides with a notion of *provability*. Such a theorem was obtained for the predicate calculus of first order with identity in 1930 by Gödel. A stronger result holding for

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2. For a brief history of this research, see Bibliographical Note below, pp. 144–45.

sets of formulas of the predicate calculus was obtained by Henkin in 1947.

My own results are analogues of Henkin's theorem for certain modal systems. There are many systems for which such results have been obtained, but in this presentation I will confine myself to three systems which seem of particular interest—two of them with completeness theorems. (The other results can, for the most part, be obtained from these two by minor changes.) The theorems themselves are mathematical in character, and are proved by algebraic techniques resembling those used in representation theorems. But when discussed in an informal way these results, so far from appearing technical, take on a metaphysical character. The reason for this is that the heuristic notions arising in the semantical interpretation of quantified modal logic are concepts of traditional philosophic concern; and so, in much the same way that a mathematician explaining his work might refer to space and its characteristics, I will be speaking of existence, substances, and accidents.

2. The central task of this paper is to provide an adequate semantical account of a sufficiently rich formal system of modal logic.

By an *adequate* semantical account, I mean one that is intuitively plausible (and hence, will provide an informally satisfying explanation of what modality is about), as well as technically acceptable. A crucial technical test of adequacy is the establishment of a semantical completeness theorem of the sort described above; but symbolic logic, being a well-articulated discipline, also imposes numerous criteria which any well-made logical theory must meet. So, as well as shaping logical work in the light of heuristic philosophical considerations, I have often found myself modifying metaphysical preconceptions in the light of technical considerations.

By a *sufficiently rich* formal system of modal logic I mean, for present purposes, something quite specific; the system must

contain formal apparatus sufficient to express the following:

Truth-functional connectives,
Modal connectives,
Identity,
Universal quantification over individuals,
Definite descriptions.

As we will see, profound difficulties are involved in handling all of these simultaneously, which cannot be settled on a piecemeal basis; for instance, one might possess a satisfactory semantics for all of the above notions except for identity, and yet have no idea of how the theory could be satisfactorily extended to account for identity.

Accordingly, our formal language will include individual variables x, y, z, ...; individual constants a, b, c, ...; i-ary ($i \geq 0$) predicate letters P, Q, ...; and sentential connectives $\supset$, $\sim$, and $\Box$. We may also construct identity-formulas $s = t$, universal quantifications $(x)A$, and descriptions $\imath_x A$. *Terms* t and *formulas* A are defined inductively as usual. By means of these primitives, we can then define disjunction, conjunction, equivalence, possibility, existential quantification, and unique-existential quantification ($\vee$, $\wedge$, $\equiv$, $\Diamond$, $\exists$, $\exists!$). We also use the following definitions:

$$\mathrm{E}t =_{\mathrm{df}} (\exists x)x = t$$

$$\mathrm{E}\Box t =_{\mathrm{df}} (\exists x)\Box(x = t).$$

The next three sections discuss some familiar ways of handling the sentential connectives, identity, and universal quantification. I will assume that the reader is familiar with these topics, and will be very brief.

3. For definiteness, let's choose a simple axiomatization of classical sentential logic, consisting of the three axiom-schemes

$$A \supset (B \supset A)$$
$$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$
$$(\sim A \supset \sim B) \supset (B \supset A),$$

and the rule *Modus Ponens.*

$$\frac{A \quad A \supset B}{B}$$

All of the systems considered below will possess these axiom-schemes and this rule in unrestricted form.[3]

The connectives $\sim$ and $\supset$ have the usual two-valued truth-functional semantics, which is characteristic for the above axiomatization; i.e. there is semantical completeness here.

The connective $\Box$ will be interpreted according to Kripke's semantics. A *model structure* consists of a nonempty set $\mathscr{K}$ (of "possible worlds") and a binary relation $\mathscr{R}$ (of "alternativeness" or "relative possibility") on $\mathscr{K}$. A formula $\Box A$ takes the value T in a member α of $\mathscr{K}$ if and only if A takes the value T in all members β of $\mathscr{K}$ such that $\alpha\mathscr{R}\beta$. By imposing various conditions on $\mathscr{R}$, we obtain interpretations corresponding to various kinds of modality: e.g., if $\mathscr{R}$ is assumed reflexive and transitive, the semantics corresponds to Lewis' **S4**.

For definiteness, the modality of **S4** will be paradigmatic in this paper; thus, I will posit the axiom-schemes

$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$

$$\Box A \supset A$$

$$\Box A \supset \Box\Box A,$$

and the rule of necessitation,

$$\frac{A}{\Box A}$$

3. One exception: the alternative to Russell's theory of descriptions mentioned in n. 8 would involve certain restrictions on the use of sentential axioms containing descriptions. But this alternative is only mentioned in passing.

in unrestricted form in the systems considered below. But the results of this paper will not depend on our choice of **S4**; they hold for most familiar systems of sentential modality.

I will refer to the axioms and rules of this section as sentential axioms and rules.

4. For the semantical interpretation of identity we introduce the notion of a *domain* of individuals; terms are assigned values in this (nonempty) domain, and $s = t$ takes the value T if and only if s and t are assigned the same member of the domain.

Let $A^{s}/\!/t$ be any result of replacing any number of free occurrences of t in A by occurrences of s (relettering bound variables, if necessary, to avoid binding a free variable in A). Then the two schemes

$$t = t \qquad \text{(SId)}$$
$$s = t \supset (A \supset As/\!/t) \qquad \text{(InId)}$$

yield an axiomatic basis for the classical theory of identity. (The second scheme will be restricted in some of the systems presented below.)

5. According to the classical theory of universality a formula $(x)A$ takes the value T if A is true of all members of the domain. This can be made suitably precise by speaking of an *interpretation* I on a domain D; such an interpretation assigns values I(x) and I(a) in D to individual variables and constants x and a, and relations I(P) on D to predicate letters P. Where d is a member of D, let $\mathrm{I}^{\mathrm{d}}/x$ be the interpretation like I except that $\mathrm{I}^{\mathrm{d}}/x(x) = \mathrm{d}$. Then:

$$\mathrm{I}((x)A) = \mathrm{T} \text{ iff for all } \mathrm{d} \in \mathrm{D}\ \mathrm{I}^{\mathrm{d}}/x(A) = \mathrm{T}.$$

Corresponding to this semantics, we have axiom-schemes

$$(x)(A \supset B) \supset ((x)A \supset (x)B) \qquad \text{(Dist)}$$
$$(x)A \supset A^{t}/x \qquad \text{(Inst)}$$

where A^{t}/x is the result of replacing all free occurrences of x

in A by occurrences of t (again, relettering if necessary), and a rule of *conditional universalization*,

$$\frac{A \supset B}{A \supset (x)B}$$

where x does not occur free in A.

6. We have not yet considered the theory of descriptions, but let us adopt for the time being Russell's theory. According to this way of handling descriptions, expressions $\iota_x A$ are not part of the primitive notation of the language; hence, the only terms are individual variables and constants. Descriptions are introduced by means of a contextual definition, of which the following is an instance:

$$P(\iota_x Q(x)) =_{df} (\exists x)((y)(Q(y) \equiv y = x) \wedge P(x)).$$

Since descriptions are defined, they may be neglected in formulating a semantics for the language, leaving truth-functions, modality, identity, and universality to be considered.

By a **Q1**-*model structure* (**Q1**ms), we will understand a system $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$, where $\langle \mathscr{K}, \mathscr{R} \rangle$ is an **S4**-model structure and $\mathscr{D}$ a nonempty domain. A **Q1**-*interpretation* I assigns a value $I(a)$ or $I(x)$ in $\mathscr{D}$ to each individual constant or variable, respectively, of the language, and to each n-ary predicate letter P a subset $I_\alpha(P)$ of $\mathscr{D}^n$ for each $\alpha \in \mathscr{K}$ (if $n = 0$, $I_\alpha(P)$ is one of the values T and F). Where I is a **Q1**-interpretation on $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ and $d \in \mathscr{D}$, let I^d/x be the **Q1**-interpretation which assigns x the value d and in other respects is just like I.

The truth-value $I_\alpha(A)$ of A in α under a **Q1**-interpretation I on a **Q1**ms $\langle \mathscr{K}, \mathscr{R}, \mathscr{D} \rangle$ (where $\alpha \in \mathscr{K}$), is defined inductively as follows:

(1) $I_\alpha(P(t_1 \ldots t_n)) = T$ if $\langle I(t_1), \ldots, I(t_n) \rangle \in I_\alpha(P)$,
$I_\alpha(P(t_1 \ldots t_n)) = F$ otherwise;
(2) $I_\alpha(s = t) = T$ if $I(s) = I(t)$,
$I_\alpha(s = t) = F$ otherwise;

(3) $I_\alpha(A \supset B) = T$ if $I_\alpha(A) = F$ or $I_\alpha(B) = T$,
$I_\alpha(A \supset B) = F$ otherwise;
(4) $I_\alpha(\sim A) = T$ if $I_\alpha(A) = F$,
$I_\alpha(\sim A) = F$ otherwise;
(5) $I_\alpha(\Box A) = T$ if for all $\beta \in \mathscr{K}$ such that $\alpha \mathscr{R} \beta$, $I_\beta(A) = T$,
$I_\alpha(\Box A) = F$ otherwise;
(6) $I_\alpha((x)A) = T$ if for all $d \in \mathscr{D}$, $I^d/x_\alpha(A) = T$,
$I_\alpha((x)A) = F$ otherwise.

(Clauses 3, 4, and 5, of this definition are standard, and will not be repeated in later definitions of satisfaction.)

This definition determines in the usual way notions of *satisfiability*, *validity*, *simultaneous satisfiability*, and *implication*, which we will term **Q1**-*satisfiability*, etc.

The question now arises, how to axiomatize the notion of **Q1**-validity. First, it is readily confirmed that all of the axioms we have presented above for truth-functional and modal connectives, for identity and for quantifiers, are **Q1**-valid; also, that all the rules presented above preserve **Q1**-validity. It therefore seems reasonable to choose at least these axioms and rules: namely, the sentential axioms and rules, together with SId, InId, Dist, Inst, and the rule of conditional universalization. These axioms and rules, however, turn out to be **Q1**-incomplete. First, although

$$\Box(x)A \supset (x)\Box A \qquad (\text{CvBP})^4$$

can be obtained from them, the **Q1**-valid formula

$$(x)\Box P(x) \supset \Box(x)P(x)$$

cannot be thus obtained. Therefore, we must adopt another

4. A note on nomenclature: 'BP' is for 'Barcan Principle,' and 'CvBP' for 'Converse Barcan Principle.' NecId asserts that identicals are *necessarily identical*, and NecDif that different things are *necessarily different*.

axiom-scheme,

$$(x)\Box A \supset \Box(x)A \qquad \text{(BP)}.^{5}$$

Second, though

$$s=t \supset \Box s=t \qquad \text{(NecId)}$$

can be obtained from InId and the other axioms, the **Q1**-valid formula

$$\Diamond a=b \supset a=b$$

cannot be thus obtained, and we therefore require still another axiom-scheme,

$$\Diamond s=t \supset s=t \qquad \text{(NecDif)}.^{6}$$

We will call the system constituted by these axioms and rules **Q1** (in a wider context, the name '**S4Q1**' would be more appropriate). In the usual way, notions of **Q1**-*theoremhood*, **Q1**-*deducibility*, and **Q1**-*consistency* are determined by the system. As it turns out, these syntactically defined concepts are equivalent in extension to the corresponding semantical concepts; we thus have semantic completeness for **Q1**, in the strong as well as the weak sense.

7. The ontology corresponding to **Q1** is reminiscent of Leibniz. The world consists of a number of substances (the members of $\mathscr{D}$) which persist through changes (conceiving of the relation $\mathscr{R}$, for the time being, as temporal), in which they may exchange attributes. These substances are never generated

5. As we will see, the import of BP is that nothing comes into existence; the import of CvBP is that nothing passes out of existence.

6. Intuitively, the content of NecDif is that things which are distinct in a given situation can never become identical. This suggests a way of interpreting identity so that NecDif would not be valid, by providing for the "merging" of distinct individuals (but *without* permitting the "separation" of identicals). If carried out, this yields a completeness theorem for the remaining axioms and rules, besides establishing the independence of NecDif.

or destroyed; their number is fixed and immutable. When, on the other hand, $\mathscr{R}$ is taken to be a relation of *metaphysical* possibility[7] (and the members of $\mathscr{K}$ are, accordingly, metaphysically possible worlds rather than temporal stages), this corresponds to the two principles that every possible object is existent, and that every existent object is necessarily existent.[8] When possibility is understood in this way, Leibniz holds that there are possible objects that do not exist in all worlds; in fact, he asserts that there are no possible objects which exist in more than one world. **Q1** may therefore be regarded as corresponding to Leibniz' notion of temporal possibility, though not to his notion of metaphysical possibility.

More generally, **Q1** is reminiscent of any atomistic ontology (ancient or modern), in which "atom" is construed in the strict sense: no atom can be generated or destroyed. The various possible worlds in $\mathscr{K}$ will correspond in this case to possible configurations of atoms; here, **S5** would be a more appropriate sentential modality, unless $\mathscr{R}$ is understood temporally and

7. I should mention at this point that I am conceiving of the notion of a model structure (and hence of a possible world), as used in the semantics of modal logic, as an abstract or structural concept having a number of kinds of realizations. (This is true in general of semantical notions—e.g. of the concept of a domain in classical quantification theory; but the abstractness is rather trivial in this case since domains have no structure.) For instance, possible worlds may be interpreted temporally, metaphysically, linguistically (as "state descriptions") or probabilistically (as sample points in a probability space). In the present paper, I will appeal mostly to the temporal interpretation, since it is the one which seems to be most valuable heuristically where quantification theory is concerned; we have some idea of how to identify individuals across temporal worlds, since we have an idea of what temporal change is like. But "metaphysical change" is more problematic.

8. The reason for this is that CvBP guarantees that since necessarily everything (i.e. every *existing* thing) exists, everything necessarily exists; while BP guarantees that if everything necessarily has a property, then nothing can come into being which does not have that property. For further information on this point, see the discussion of **Q3** below, sections **14** ff.

physical laws are not required to be symmetric with regard to time.

Further insights into **Q1** are gained by considering the solution which it affords to the following stock modal paradox:

$$\Box(2 < 3)$$
$$3 = \text{the number of living Presidents}$$
$$\therefore \Box(2 < \text{the number of living Presidents}).$$

In general, there are a number of ways of handling such a paradox: denying either of the premises, denying the validity of the inference but holding that the conclusion (though apparently false) is innocuous, or holding that any of the statements in the argument is meaningless. **Q1** takes the third course; though it sanctions the inference, the conclusion, once formalized according to the Russellian theory of descriptions, has the form

$$(\exists x)[(y)(NP(y) \equiv x = y) \wedge \Box 2 < x],$$

which is harmlessly true. The paradox is thus *explained away*, since despite appearances the conclusion does not turn out to be a statement of necessity.

8. Two objections to **Q1** are suggested by the above considerations: first, that **Q1** supposes that the same individuals are found in all possible worlds, and second, that **Q1** must employ Russell's theory of descriptions. Perhaps more should be said at this point concerning the second of these objections, since all we have claimed so far is that this theory explains away our Presidential paradox; but surely, explaining away is not in itself objectionable. Quite apart from this, however, it seems to me that the Russellian theory is infected with inelegancies and inadequacies. The inelegancies result from the unperspicuous notation and awkward considerations of scope arising from the contextual definition of descriptions. An inadequacy closely related to this is that the formalization

of sentences from natural languages is *artificial* in many cases; e.g. 'The largest prime number is even or not even' is formalized not as an instance of excluded middle, but as an existential quantification. Another inadequacy is that it is plausible to regard some sentences about nonexistents as true; e.g. sentences asserting their nonexistence.[9] All of these objections to the Russellian theory are interrelated in such a way that a proponent of Russell's theory can meet them by making distinctions of scope; but a dialogue of this sort soon shows that this theory results in strained relationships to natural language. The Russellian must eventually cut his moorings and claim that natural language is inherently "confused" or "vague."

It is significant that both of the above sources of dissatisfaction with **Q1** point in the same direction: to a revision of classical quantification theory as regards existential presuppositions. In particular, the principle called into question is Inst; when contraposed, this yields the principle

$$A^{t}/x \supset (\exists x)A \qquad \text{(CvInst)}.$$

And the special case

$$t = t \supset Et$$

guarantees that every term will refer. But this is clearly incompatible with allowing names to designate objects not in the actual world (e.g. objects which did exist, but don't at this time), as well as with allowing descriptions to be primitive. Both modifications force us to give up the assumption that individual terms always refer.

Now, systems of quantification theory modified in this way have been proposed and investigated during the past ten years

9. The first two difficulties can be resolved by allowing descriptions to be primitive, and assigning any formula the value F if it contains a description not meeting the unique-existence condition; if the condition is met, the description is assigned the obvious designatum. This semantical theory of descriptions is readily axiomatized and proved complete; but this involves revising classical sentential logic, and does not meet our third objection to the Russellian theory.

or so by a number of logicians,[10] and their semantical and syntactical characteristics disclosed. As far as the predicate calculus without modality and descriptions is concerned, at least two semantical interpretations have been proposed: one by van Fraassen, and one by Leblanc and myself. Although van Fraassen's theory is better adapted to description theory, I will confine my remarks here to the latter of these interpretations, which is easier to formulate. According to this interpretation there are *two* (disjoint) domains: a nonempty "inner" domain D and an "outer" domain D′. Individual terms are assigned values in either domain, and predicate letters assigned relations on the union of the domains; thus $P(a)$, e.g. will be either true or false under any interpretation even if a is assigned a "nonexistent" value, in the outer domain. Bound variables are construed as ranging over the inner domain, so that $\mathrm{I}((x)A) = \mathrm{T}$ if for all $\mathrm{d} \in \mathrm{D}$, $\mathrm{I}^{\mathrm{d}}/x(A) = \mathrm{T}$.

Syntactically, the formulas valid under this interpretation may be characterized by axioms and rules sufficient to generate all tautologies, together with the rule of conditional universal generalization, the axiom-schemes InId, SId, Dist, and the following axiom-schemes:

$$(x)A \supset (\mathrm{E}t \supset A^t/x) \qquad \text{(EInst)}$$
$$(\exists x)\mathrm{E}x \qquad \text{(NEm)}$$
$$(x)\mathrm{E}x \qquad \text{(EU).}$$

Following Lambert's terminology, we call this logic *free* quantification theory; the system determined by the above axioms and rules will be called **F**.

9. Free quantification theory is to be used as the basis of a more natural theory of descriptions, and of a modal semantics allowing the generation and destruction of individuals. I will first turn to the theory of descriptions. The account I will

10. Lambert, Hintikka, Hailperin and Leblanc, van Fraassen, and myself, among others.

describe is similar to one due to van Fraassen; he and Lambert also have a completeness theorem for it.[11] According to this theory, the value $\mathrm{I}(\imath_x A)$ assigned by I to $\imath_x A$ is the unique member d of the inner domain, such that $\mathrm{I}^{\mathrm{d}}/x(A) = \mathrm{T}$, if there exists such a unique member; otherwise, $\mathrm{I}(\imath_x A)$ is an arbitrary member of the outer domain (in description theory, then, we must assume the outer domain to be nonempty).

It turns out that only two simple and plausible axioms are needed to suit this interpretation; these, in conjunction, of course, with **F**, yield a *minimal* kind of description theory.

$$\mathrm{E}\imath_x A \supset (\exists! x)A \qquad \text{(DE)}$$
$$(x)[(y)[A^y/x \equiv x = y] \supset x = \imath_x A] \qquad \text{(DI)}$$

DI is subject to the restriction that y have no free occurrences in A.

10. We may now proceed to sketch a semantical interpretation of the whole language: truth-functions, modality, universal quantification, identity, and descriptions. The resulting theory **Q2** will play a dialectical role in the present paper, serving to introduce problems which (I shall claim) are resolved by the theory **Q3**. Anyone who tries to verify whether particular formulas are valid in **Q2** will soon find that this notion of validity is very complicated. As it turns out, this complexity is more than just apparent; David Kaplan has informed me that there is no axiomatization of **Q2**—the set of **Q2**-valid formulas is not recursively enumerable. (This result is unpublished, and depends on work which appears in Kaplan's dissertation.) With this in mind, let's turn to a semantical characterization (which, of course, will be infinitistic) of **Q2**.

11. Clearly, our semantical interpretation will involve a system of possible worlds, each with a domain. Besides these.

11. They differ from the ones given in my "Some Completeness Results for Modal Predicate Calculi," Proceedings of the 1968 Irvine Philosophy Colloquium, forthcoming.

we will want still another set to serve as an outer domain, since otherwise every singular term (including, e.g. $\imath_x(P \wedge \sim P)$) would perforce be assigned an individual existing in some possible world.

Individual terms are assigned values in the domains of these possible worlds. When we did this in connection with **Q2**, we assigned each term a value which was fixed beforehand for all worlds; the value assigned to a in α was the same as the value assigned to a in β. This procedure seemed natural at the time, because we had supposed that the same domain was associated with all worlds; but this assumption glosses over the problem of *identifying individuals across worlds*.

In the present more general case, then, we define a **Q2**-*model structure* $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}' \rangle$ in such a way that the component $\mathscr{D}$, rather than being a fixed domain, is a *function* taking worlds α into nonempty domains $\mathscr{D}_\alpha$. $\mathscr{D}'$ is a nonempty domain serving as a "limbo" for individuals not existing in any world in $\mathscr{K}$. As before, we suppose that $\langle \mathscr{K},\mathscr{R} \rangle$ is an **S4** model structure.

Before determining how to assign values to terms in general, let us consider a description: say, 'the thing in place p.' We fix the designation of this term in a given world according to whether there is a unique individual in the domain of that world having the property of being in place p in that world; if there is, we assign that thing to the term. But individuals can change position, and in general the designatum of $\imath_x P(x)$ in world α and the designatum of $\imath_x P(x)$ in world β will not be the same.

Applying this idea to terms in general, let us regard a **Q2**-interpretation, I, on a **Q2** ms $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}' \rangle$ as assigning for each $\alpha \in \mathscr{K}$ a value $I_\alpha(t)$ in $\mathscr{D}' \cup \bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha$ to each individual variable or individual constant t; and to each description $\imath_x A$ a value $I_\alpha(\imath_x A) = d$, where if there is a unique member e

of $\mathscr{D}_\alpha$ such that[12] $I^e/x(A) = T$, then d = e, and if there is no such member, then $d \notin \mathscr{D}_\alpha$; and finally, to each n-ary predicate letter P a relation $I_\alpha(P)$ on $\mathscr{D}' \cup \bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha$. Predicate letters, then, are assumed to be defined for all individuals: those in $\mathscr{D}'$, as well as those in the domains of the possible worlds. In this way, excluded middle is preserved.

The definition of **Q2**-*satisfaction* is straightforward in the case of atomic formulas:

(1) $I_\alpha(P(t_1 \ldots t_n)) = T$ if $\langle I_\alpha(t_1), \ldots, I_\alpha(t_1) \rangle \in I_\alpha(P)$,
$I_\alpha(P(t_1 \ldots t_n)) = F$ otherwise;
(2) $I_\alpha(s=t) = T$ if $I_\alpha(s) = I_\alpha(t)$,
$I_\alpha(s=t) = F$ otherwise.

An atomic formula, e.g. $P(a)$, is true in a possible world if and only if the relation assigned to P in that world holds of the individual assigned to a in that world.

Cases (3), (4), and (5) of the definition are treated as on p. 125, above. Using only these clauses of the definition, it may be seen that

$$s=t \supset (A \supset A^s/\!\!/t)$$

is valid provided that no occurrence of t in A at which an occurrence of s appears in $A^s /\!\!/ t$ is within the scope of a modal operator (in other words, provided that s is substituted for t only in *transparent* contexts). Let us call this qualified form of InId "RInId" ("Restricted Indiscernability of Identicals"). Although RInId is **Q2**-valid, InId is not. E.g. let $\mathscr{D}_\alpha = \mathscr{D}_\beta = \{0,1\}$, where $\alpha \neq \beta$, and $\alpha\mathscr{R}\beta$, and let $I_\alpha(a) = I_\alpha(b) = 0$ and $I_\beta(a) = 0$, $I_\beta(b) = 1$. Then $I_\alpha(a=b \supset (\Box a=a \supset \Box a=b)) = F$ (and $I_\alpha(a=b \supset \Box a=b) = F$ as well). The principle

$$\Box s=t \supset (A \supset A^s/\!\!/t) \qquad (\text{In}\Box\text{Id}),$$

12. Strictly speaking, this is sloppy. Actually, **Q2**-*interpretation* and *truth value under a* **Q2**-*interpretation* should be defined by a simultaneous induction.

however, is **Q2**-valid. It is clear, therefore, that in **Q2** the Presidential paradox (above, p. 128) is solved by denying the validity of the argument, which depends on the principle InId. The force of the paradox may then be ascribed to the failure to distinguish identity from necessary identity, with resultant confusion of InId with In□Id.

All of this may be seen without completing the definition of satisfaction; at some point, however, we must determine the truth-conditions of universal formulas. When A contains no modal operators, it is clear that $\mathrm{I}_\alpha((x)A)$ should be T if and only if for all d in $\mathscr{D}_\alpha$, $\mathrm{I}'_\alpha((x)A) = \mathrm{T}$ for all interpretations I′ differing from I at most in the value assigned to x in α, and such that $\mathrm{I}'_\alpha(x) \in \mathscr{D}_\alpha$. But, e.g. in the case of $(x)\Diamond P(x)$, the truth-value in α will depend in general not only on the interpretation of x in α, but on the values assigned to x in other worlds as well. How, then, are we to define the truth-value in such cases?

There are some radical approaches to this problem which merit brief discussion. First, one could restrict the language in such a way that well-formed formulas do not involve any quantification into modal contexts. When bolstered by philosophic arguments, this approach may appear plausible, but at bottom these arguments simply restate the fact that there is a problem. Thus, this seems to me to be more a suppression than a solution of the difficulty; certainly, if this technique were generally applied to areas in which difficult problems arise, life would be rather dull.

Another solution[13] would be to interpret quantification as having to do with the substitution of names; we could then say that $\mathrm{I}_\alpha((x)A) = \mathrm{T}$ if for all terms t of the language, $\mathrm{I}_\alpha(A^t/x) = \mathrm{T}$. This proposal raises a host of technical difficulties. Obviously, it conflicts with the decisions made above concerning existence, since Inst is valid under this interpretation. Even if this and related difficulties were solved, a more vexing problem would arise, concerning descriptions. The definition of satisfaction under this interpretation of quantification presupposes that the truth values of *all* identities are fixed beforehand, but the definition of identities involving descriptions presupposes in turn the definition of satisfaction. This circularity is *vicious*: the usual sort of inductive definition breaks down, and there is no straightforward way of guaranteeing that a satisfaction function I exists, given an assignment of values to atomic formulas and atomic terms. Of course, this problem could be solved by appealing to Russell's theory of descriptions or a similar theory; but this limitation is surely a defect. Further technical difficulties arise concerning the definition of implication, since under the substitution interpretation the rule of complete induction is valid. It therefore follows from Gödel's incompleteness theorem (or, equally well, from Tarski's theorem) that any effective formulation of this quantification theory will be incomplete as to consequences, since all of the truths, e.g. of first-order arithmetic, would be implied by the usual axioms; but the class of arithmetic truths is not effectively enumerable. To top these difficulties, there is the more philosophical objection that the substitution interpretation is plausible as an interpretation of *quantification* only if everything has a name. But this last condition is not always met, and there does not seem to be any good reason to suppose in logic that it is. For these reasons, though I feel that the substitution interpretation (i.e. the rule of complete induction) is worthy of investigation in its own right, it does not seem to me that substitution is a satisfactory surrogate for quantification over individuals.

The most attractive remaining alternative—certainly, the one most in harmony with the semantic approach of this paper—is the one in which formulas of the sort $(x)A$ are

13. This has been suggested by R. Barcan Marcus.

interpreted universally with respect to *all* ways of identifying the value of x. We make this more precise by letting a *world-line* d on a **Q2**-model structure $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}'\rangle$ be any function from $\mathscr{K}$ into $\mathscr{D}'' \cup \bigcup_{\alpha\in\mathscr{K}} \mathscr{D}_\alpha$. Also, let I^d/x differ from I in that for all $\alpha \in \mathscr{K}$, $I^d/x_\alpha(x) = d_\alpha$. The final clause of **Q2**-*satisfaction* then reads as follows:

(6) $I_\alpha((x)A) = T$ if for all world-lines d on $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}'\rangle$ such that $d_\alpha \in \mathscr{D}_\alpha$, $I^d/x_\alpha(A) = T$,
$I_\alpha((x)A) = F$ otherwise.

The only technical difficulty with this definition is that the class of world-lines may be nondenumerably infinite, even in cases where $\mathscr{D}'$ and the $\mathscr{D}_\alpha$ are denumerable; but this will seem a glaring defect only to those who do not countenance non-denumerable infinities. More problematic is the fact that certain rather counterintuitive theses turn out to be validated in **Q2**. An example of this sort is

$$(\exists x)\Box Ex;$$

this ontologically tantalizing conclusion follows from the more general principle

$$\Box(\exists x)A \supset (\exists x)\Box A,$$

where no free occurrence of x in A falls within the scope of a modal operator—which also is **Q2**-valid.

12. Though I believe **Q2** and related logics to be worthy of further investigation, I take these results to be symptomatic of a deeper infirmity, having to do with the notion of *substance*. Recall that above, when we were motivating the sort of reference involved in **Q2**, we used descriptions as a paradigm. The effect of this was to suggest that the fundamental way of identifying individuals across worlds was by means of *properties* (e.g. we identify the thing in place p by means of a distinguishing property), and this in turn suggested that all world-lines are

semantically on a par. When all such identifications are subsumed under this paradigm, one is led to the conclusion that, e.g. Socrates-at-t_1 is identified with Socrates-at-t_2 (i.e. shares with Socrates-at-t_2 a world line which is a value of a variable), because the properties of the one are like the properties of the other. And the slipperiness of this "likeness" soon suggests that it is arbitrary or conventional that these are given the same name: any world-line is, in reality, as worthy to be the value of a variable as any other.

In contrast to the view sketched in the above paragraph, it seems to me that we do—at least, in temporal cases—identify individuals across worlds in a way more absolute than that taken into account by **Q2**. Thus, I am inclined to take the sort of reference in which a name ('Socrates') is assigned *one thing* (Socrates) *which is the same in many possible worlds*, as primary or paradigmatic. The defect, then, of **Q2**, is that it does not allow for this unity: for preferred world-lines which may be regarded as single things remaining fixed through a change. In a word, **Q2** lacks a concept of *substance*.

13. Before formulating a system **Q3** in accordance with these suggestions, it may be appropriate to support the above opinions by a philosophical consideration or two. It may be that those who held views similar to those motivating **Q2** were led to do so by the impression that identification of *properties* across worlds is somehow less problematic than identification of *individuals* across worlds; this, certainly, would be a reason to attempt the elimination in modal logic of singular terms in favor of distinguishing properties. But there is, so far as I can see, absolutely no reason to suppose that this is the case: if anything, the identification of properties is *more* problematic. How can properties be identified across worlds (without, of course, appealing to substances)?

Certainly, not by means of their extension, for this pre-supposes identification of individuals. Perhaps, then, by means of properties of properties; but this obviously leads to an infinite regress. If there is reason to cut this regress anywhere in order to allow a foundation for the whole series, it is at the very beginning: namely, with identification of individuals. Finally, perhaps by intensional criteria; but these would have to be explicated. If the explication were to involve the notion of alternative situations or possible worlds, the problem would remain unsolved; but it is hard to imagine an adequate explication that would not involve this notion in some form.[14]

These arguments are not without double edges. For instance, if I were pressed as to how individuals are identified and give any sensible answer (say, "by continuity"), I could be led in similar circles. But I do feel that such arguments can legitimately be used to remove prejudices or misconceptions that may stand in the way of **Q3**.

14. As in the case of **Q2**, a **Q3**-*model structure* is a quadruple $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}' \rangle$. Now however, we will construe overlaps of the $\mathscr{D}_\alpha$ as indicative of substances; i.e. we regard an individual as existing in both α and β if it is a member of $\mathscr{D}_\alpha \cap \mathscr{D}_\beta$. More precisely, we will regard all individuals in $\mathscr{D}' \cup \bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha$ as *already* identified across worlds; any individual d of a **Q3**ms is the same with respect to all worlds of the model structure. We will construe individual variables as ranging over substances, and thus will take seriously the classical doctrine that

14. To give this argument some content, imagine a world possible to this one in which everyone calls a horse's tail a leg. What we identify in that world with the property of being a horse's leg will depend on whether we go on the principle that "calling a tail a leg makes it one," or the principle that "a tail isn't a leg, whatever you call it." (I have been told that this elegant example goes back to Buridan, but have not verified the reference.)

Strictly speaking, the problem of identifying properties across worlds does not arise explicitly until one attempts to provide a semantics for *second-order* modal logic. Nevertheless, I would claim that it is present implicitly in the modal notion of *property*.

only substances are "beings" in the fullest sense of the word. In practice, this means that if an individual variable is assigned a value in world α, it is automatically assigned the same value in every other world β (whether or not it is in the domains of these worlds). Thus, so far as the interpretation of individual variables goes, we are reverting to the sort of reference we had in **Q1**.

As regards descriptions, however, we will retain the treatment of **Q2**; this means that descriptions may refer to non-substances, even though the individual variables range in **Q3** over substances; there is thus a major difference in the way **Q3** treats individual variables, on the one hand, and descriptions, on the other. But this is precisely the distinction that is wanted, and creates no difficulties since the underlying quantification theory is free; descriptions need not designate values of individual variables.

The question of how to handle individual constants is delicate. Most uses of proper names seem to be of the sort in which substances are intended; e.g.

$$(x)(x = \text{Socrates} \supset \Box x = \text{Socrates})$$
$$\text{and } (x)(x = \text{Texas} \supset \Box x = \text{Texas})$$

have a ring of truth. This would suggest that in **Q3** individual constants should be treated like individual variables. There are, however, some uses of proper names (roughly classifiable as *titulary* uses) which do not meet this condition. For example, if we call anyone holding a certain political office "Caesar," then various people may be Caesar in different possible worlds, and

$$(x)(x = \text{Caesar} \supset \Box x = \text{Caesar}),$$

or equivalently,

$$E\Box\text{Caesar}$$

is false. Similarly with "Miss America," and other cases in which a name is *won*. "Coriolanus" is perhaps a borderline case. It seems rather artificial to make a *syntactical* distinction between these and other sorts of namings, and therefore in **Q3** I will treat individual constants more like descriptions than individual variables; any world-line may be assigned to an individual constant. The supposition that an individual constant refers to a substance can, however, always be made explicitly, by means of assertions of the sort

$$\mathrm{E}\Box a.$$

15. A **Q3**-*interpretation* on a **Q3**ms $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}' \rangle$, then, is an assignment I of members of $\mathscr{D}' \cup \bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha$ to individual variables, world-lines on $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}' \rangle$ to individual constants, and relations on $\mathscr{D}' \cup \bigcup_{\alpha \in \mathscr{K}} \mathscr{D}_\alpha$ to predicate letters (one for each world in $\mathscr{K}$). This may be summarized by saying that a **Q3**-interpretation is a **Q2**-interpretation, except that for all individual variables x and all $\alpha, \beta \in \mathscr{K}$, $\mathrm{I}_\alpha(x) = \mathrm{I}_\beta(x)$. Descriptions are handled precisely as in **Q2**.

Let $\mathrm{I}^{\mathrm{d}}/x$ differ from I at most in that for all $\alpha \in \mathscr{K}$, $\mathrm{I}^{\mathrm{d}}/x_\alpha(x) = \mathrm{d}$. Then the most important clauses of the definition of **Q3**-satisfaction are:

(1) $\mathrm{I}_\alpha(P(t_1 \dots t_n)) = \mathrm{T}$ if $\langle \mathrm{I}_\alpha(t_1), \dots, \mathrm{I}_\alpha(t_n) \rangle \in \mathrm{I}_\alpha(P)$,
$\mathrm{I}_\alpha(P(t_1 \dots t_n)) = \mathrm{F}$ otherwise;
(2) $\mathrm{I}_\alpha(s = t) = \mathrm{T}$ if $\mathrm{I}_\alpha(s) = \mathrm{I}_\alpha(t)$,
$\mathrm{I}_\alpha(s = t) = \mathrm{F}$ otherwise;
(6) $\mathrm{I}_\alpha((x)A) = \mathrm{T}$ if for all $\mathrm{d} \in \mathscr{D}_\alpha$, $\mathrm{I}^{\mathrm{d}}/x_\alpha(A) = \mathrm{T}$,
$\mathrm{I}_\alpha((x)A) = \mathrm{F}$ otherwise.

16. In seeking an axiomatization of **Q3**, we must first note that the principle EInst is **Q3**-invalid. To **Q3**-satisfy, e.g. the negation of

$$(x)\Box \mathrm{E}x \supset (\mathrm{E}a \supset \Box \mathrm{E}a)$$

(an instance of EInst), let $\mathscr{D}' = \{0\}$, and $\mathscr{D}_\alpha = \mathscr{D}_\beta = \{1\}$, where $\alpha \neq \beta$ and $\alpha \mathscr{R} \beta$. Then let $\mathrm{I}_\alpha(a) = 1$ and $\mathrm{I}_\beta(a) = 0$. Now,

$\mathrm{I}_\alpha((x)\Box \mathrm{E}x) = \mathrm{T}$ (because $\mathscr{D}_\alpha \subseteq \mathscr{D}_\beta$), and $\mathrm{I}_\alpha(\mathrm{E}a) = \mathrm{T}$; but $\mathrm{I}_\beta(\mathrm{E}a) = \mathrm{F}$, and so $\mathrm{I}_\alpha(\Box \mathrm{E}a) = \mathrm{F}$. The reason for this failure of EInst in **Q3** is clear; the variables of **Q3** range over *substances*: i.e. over objects identified across worlds. Thus, terms which fail to refer to the same object in all worlds, as well as terms not referring to existents, may not instantiate such variables. This diagnosis indicates how to remedy the difficulty; in **Q3**, we posit the axiom-scheme

$$(x)A \supset (\mathrm{E}\Box t \supset A^t/x) \qquad \text{(SInst)}.$$

Besides SInst, we will incorporate in **Q3** the sentential axioms and rules, Dist, NEm, and EU;[15] also the principles DI and DE, and SId and RInId. Our treatment of free individual variables renders two further axiom-schemes indispensible:

$$x = y \supset \Box x = y, \text{ and } \Diamond x = y \supset x = y,$$

where x and y are individual variables.[16]

15. The system **Q3Em** obtained by dropping the Axiom-scheme EM from **Q3** is characterized semantically by relaxing the restriction on **Q3** model structures $\langle \mathscr{K},\mathscr{R},\mathscr{D},\mathscr{D}' \rangle$ that $\mathscr{D}_\alpha$ be nonempty for all $\alpha \in \mathscr{K}$.

16. In the proof of semantical completeness given in my "Some Completeness Results for Modal Predicate Calculi," five additional rules must be incorporated in **Q3**.

$$\frac{A \supset B}{A \supset (x)B}, \quad \text{where } x \text{ is not free in } A.$$

$$\frac{A \supset \Box B}{A \supset \Box(x)B}, \quad \text{where } x \text{ is not free in } A.$$

$$\frac{A \supset .. B_1 \prec . \dots \prec . B_n \prec \Box C}{A \supset . B_1 \prec . \dots \prec . B_n \prec \Box(x)C}, \quad \text{where } x \text{ is not free in } A, B_1, \dots, \text{or } B_n.$$

$$\frac{A \supset \sim t = x}{\sim A}, \quad \text{where } x \text{ is not free in } A \text{ or in } t.$$

$$\frac{A \supset . B_1 \prec . \dots \prec . B_n \prec \sim t = x}{A \supset . B_1 \prec . \dots \prec . \Box \sim B_n}, \quad \text{where } x \text{ is not free in } A, B_1, \dots, B_n, \text{or } t.$$

It is not known at present whether or not these rules are redundant.

The system **Q3** given by these axioms and rules is semantically complete, in both the weak and strong senses.

17. The above list of axioms and rules, and the definition of **Q3**-validity, in all likelihood will provide the reader with only a rough idea of what the system **Q3** is like. It may be helpful at this point to list some **Q3**-valid, **Q3**-invalid, and **Q3**-satisfiable formulas. Of course, every substitution-instance of a sentential theorem of **S4** is **Q3**-valid; so also is any substitution-instance of a theorem of classical quantification theory *without free individual terms*. E.g. any formula of the kind $(x)((y)A \supset A^{x}/y)$ is **Q3**-valid. Besides these, all formulas of the following sorts are **Q3**-valid:

$$\mathrm{E}\imath_x P(x) \supset P(\imath_x P(x)),$$
$$(y)(y=\imath_x P(x) \equiv (P(y) \wedge (x)(P(x) \equiv x=y))),$$
$$\Box(\exists x)\mathrm{E}\Box x,$$
$$\Box(x)\mathrm{E}\Box x, \text{ and}$$
$$(x)(y)(x=y \supset (A \supset A^{x}/\!\!/y)) \quad \text{InIdS}).$$

The following formulas, however, are **Q3**-invalid:

$$(x)\Box P(x) \supset \Box(x)P(x) \qquad (1),$$
$$\Box(x)P(x) \supset (x)\Box P(x) \qquad (2),$$
$$\Diamond(x)P(x) \supset (x)\Diamond P(x) \quad \text{and} \qquad (3),$$
$$a=\imath_x P(x) \supset (\Box Q(a) \supset \Box Q(\imath_x P(x))) \qquad (4).$$

The first three of these formulas may be falsified in a **Q3**-model structure with only two worlds α and β, where $\alpha\mathscr{R}\beta$. To falsify (*1*), let $\mathscr{D}_\alpha = \{0\}$ and $\mathscr{D}_\beta = \{0,1\}$, and let $\mathrm{I}_\alpha(P) = \{0\}$ and $\mathrm{I}_\beta(P) = \{0\}$. Then $\mathrm{I}_\alpha((x)\Box P(x)) = \mathrm{T}$, but $\mathrm{I}_\beta((x)P(x)) = \mathrm{F}$, and hence (*1*) is false. (Note that this is a case of *coming to be*.) To falsify (*2*), let $\mathscr{D}_\alpha = \{0,1\}$ and $\mathscr{D}_\beta = \{0\}$, and let $\mathrm{I}_\alpha(P) = \{0.1\}$ and $\mathrm{I}_\beta(P) = \{0\}$. Since $\mathrm{I}^1/x_\beta(P(x)) = \mathrm{F}$, $\mathrm{I}^1/x_\alpha(\Box P(x)) = \mathrm{F}$ and hence $\mathrm{I}_\alpha((x)\Box P(x)) = \mathrm{F}$, since $1 \in \mathscr{D}$. But, $\mathrm{I}_\alpha(\Box(x)P(x)) = \mathrm{T}$, and hence (*2*) is false. (Note that this is a case of *passing away*.) Finally, to falsify (*3*), let $\mathscr{D}_\alpha = \{0,1\}$ and $\mathscr{D}_\beta = \{0\}$, and $\mathrm{I}_\alpha(P) = \{0\}$ and $\mathrm{I}_\beta(P) = \{0\}$. Now, $\mathrm{I}^1/x_\alpha(P(x)) = \mathrm{F}$ and $\mathrm{I}^1/x_\beta(P(x)) = \mathrm{F}$, so $\mathrm{I}^1/x_\alpha(\Diamond P(x)) = \mathrm{F}$ and hence $\mathrm{I}_\alpha((x)\Diamond P(x)) = \mathrm{F}$, since $1 \in \mathscr{D}_\alpha$. But $\mathrm{I}_\beta((x)P(x)) = \mathrm{T}$, so $\mathrm{I}_\alpha(\Diamond(x)P(x)) = \mathrm{T}$; hence (*3*) is false.

The duals of (*1*)–(*3*) are, of course, also invalid.

The formula $(x)\Diamond \sim \mathrm{E}x$ is **Q3**-satisfiable, as well as the slightly more gloomy formula $(x)\Box\Diamond \sim \mathrm{E}x$. It therefore is consistent in **Q3** to suppose that everything is perishable, or even that everything is *necessarily* perishable. (It would, of course, be **Q3**-inconsistent though to assert that everything is *simultaneously* perishable.)

18. Finally, it will be worthwhile to discuss the way in which **Q3** solves the presidential paradox; like **Q2**, the system **Q3** renders the argument of the paradox—which has the form

$$a=\imath_x P(x)$$
$$\Box Q(a)$$
$$\therefore \Box Q(\imath_x P(x))$$

and is thus an instance of (*4*), above—invalid, and so blocks the paradoxical inference. But besides providing a well-articulated semantical theory in which the argument is invalidated, **Q3** disarms the paradox in a deeper and more significant sense, by at once explaining why the argument of the paradox is seductive, and at the same time fortifying our resistance to those seductions.

The argument of the paradox is plausible because it is an instantiation of InIdS (i.e. of $(x)(y)(x = y \supset (A \supset Ay/x)))$, which is **Q3** *valid*. But the semantical theory of **Q3** makes it clear that the universality of InIdS applies only to *substances*, not to nonsubstantial world-lines. It is therefore a conflation of substance and accident that makes the paradox plausible; and in retrospect it is not surprising that if this conflation is made, one should be able to generate modal paradoxes. The difficulty of this puzzle was compounded by the uncritical acceptance during the first half of our century of the principle Inst; thus, the crucial step in loosening this modal *aporia* must be credited to Leonard, Lambert, Hintikka, Leblanc, Hailperin, and other workers in the logical analysis of existence.

19. Some readers may feel that the present paper has been long on logic and short on metaphysics. It would have been possible to render the above material more philosophical and less technical, but it is my feeling that, at the present stage of philosophic inquiry into these topics, what is most needed is a firmer technical foundation; and this I have tried to provide. I hope, though, that the relevance of this discussion to traditional metaphysical concepts is clear. And—though I have not made this explicit either—its relevance to modern philosophical treatments of logical problems concerned with modality, reference, and existence will be apparent to anyone familiar with the literature on this topic.

Bibliography

Bibliographical Note: The first semantical account of modality known to me is that of Carnap [1]; this paper includes a treatment of quantifiers, which may be regarded as a special case of the theory **Q1** formulated in the present paper. During the 1950s the semantics of sentential modality was developed by several researchers, including Montague, Kanger, and Kripke. The most detailed presentation of this theory is in Kripke's article [7]; references to other semantical theories of modality (e.g. the algebraic interpretation of McKinsey and Tarski) are also given there. During this period, all of the above three authors were developing or had developed a semantical theory of quantification as well; quantification is treated explicitly in Kanger[5] and Kripke [6], and a quite sophisticated theory is sketched in Montague [11]. In the 1960s these ideas have crystalized and developed into general, rigorous semantical

theories of modal languages with quantifiers. Much of this recent research is still unpublished, and to my knowledge, the mathematical details of a general theory have not yet appeared in print. But expositions of such theories have been published during the past six years; among the most important of these are Hintikka [2] and [3], and Kripke [8].

This account is far from being historically complete; other logicians who have contributed to the subject are Dana Scott, David Kaplan, and Arthur Prior, and many more. Research in this area seems to a remarkable extent to have developed independently; my own work, though inspired by Kripke's early papers, has also developed without much influence from the above sources.

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