A semantic analysis of conditional logic[1]

by

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1. *Introduction*

Many philosophers have found conditional statements—particularly counterfactual conditional statements—both important and problematic.[2] Such statements have been considered important because questions concerning conditionals are closely linked to the explanation of law-like connections, the problems of induction and causation, the analysis of dispositional concepts, and other problems central to the philosophy of science. They have been thought problematic since they have defied plausible analysis within the framework of truth-functional logic.

Our intention in the present paper is to clarify the logical status of conditionals, not by analysis within classical logic, but by the construction of a formal system having a primitive conditional connective; besides providing an axiomatic formulation of the system, we will also present an intuitively plausible semantic theory for the conditional connective. The sentential part of this formal system was proved complete in [16], and the semantic

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[2] For early statements of some of the problems concerning subjunctive and counterfactual conditionals, see Lewis [7] (especially pp. 211—230), Goodman [5] and Chisholm [2]. For discussions of laws of nature and their relation to counterfactuals, see Nagel [8], pp. 47—78, and Pap [9], pp. 289—97; for the problem of dispositionals, see Carnap [1]. Goodman [4] maps the relationships among many of these problems and suggests a direction for their solution. Ryle [13] and von Wright [22], pp. 127—165, analyze conditionals as "inference tickets" and "modes of asserting", respectively. Rescher in [10] and [12] discusses the more general problem of reasoning from belief-contravening hypotheses.

theory was given an informal explanation and defense in [14]. In the present paper, we extend this system and completeness theorem to include quantification over individuals; in so doing, we will incorporate much of the material in [16].

Specifically, we will characterize below a basic system of first-order conditional logic with identity and prove this theory sound and complete with respect to the semantic interpretation. In subsequent sections, we will suggest various ways in which the basic system can be modified and extended, and will consider briefly some philosophical applications of the theory.

2. *Morphology*

A *morphology* M for first-order and conditional logic with identity consists of the following sets of symbols.

(1) *Logical symbols*: $\{\supset, \sim, =, >\}$.
(2) *Parentheses*: {), (}.
(3) *Individual variables*: a denumerable set V_M.
(4) *Individual constants*: a denumerable or finite set C_M.
(5) *Predicate letters*: for each nonnegative integer n a denumerable set P^n_M.

For the present, the set T_M of *terms* of M consists only of the individual variables and individual constants of M, and the set W_M of *wffs* of M is defined inductively in the usual way. The only unfamiliar clause in the definition concerns the logical symbol $>$, the *corner*: if A and B are wffs, then $(A>B)$ is a wff. Later, we will extend T_M to include definite descriptions, and W_M to include wffs involving abstracts.

We shall use 'A', 'B', 'C', 'x', 'y', 'z', 's', 't', 'P', 'Q', and 'R' as *syntactic metavariables*: the first three for wffs, the next three for individual variables, the next two for terms, and the last three for predicate letters. A^s/t is the result of replacing all free occurrences of t in A by s (relettering bound variables, if necessary, to ensure that all new occurrences of s are free), and $A^s//t$ the result of replacing zero or more free occurrences of t in A by s (relettering, as before, if necessary).

We shall use the customary contextual definitions of '$\vee$', '$\wedge$', '$\equiv$' and '$\exists$', and standard conventions for eliminating parentheses.[3] The modal expressions '$\Box A$' and '$\Diamond A$' are contextually defined as '$(\sim A>A)$' and '$\sim(A>\sim A)$', respectively.[4] The expressions 'Et' and '$E\Box t$' should be understood as abbreviations for '$(\exists x)x=t$' and '$(\exists x)\Box x=t$', where x is the alphabetically first variable differing from t.

3. *Semantics*

D3.1. A CQ *model structure* (CQms) is a structure $\mathcal{M}=\langle\mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}'\rangle$ where $\lambda\in\mathcal{K}$, $\mathcal{K}'=\mathcal{K}-\{\lambda\}$ is a non-empty set, $\mathcal{R}$ is a binary reflexive relation on $\mathcal{K}'$, $\mathcal{D}$ is a function taking members α of $\mathcal{D}'$ into possibly empty sets $\mathcal{D}_\alpha$, and $\mathcal{D}'$ is a set disjoint from $\bigcup_{\alpha\in\mathcal{K}}\mathcal{D}_\alpha$.

The notion of a model structure is taken from Kripke's semantics for modal logic. The set $\mathcal{K}'$ $(=\mathcal{K}-\{\lambda\})$ should be understood as the set of *possible worlds*, or (following D. Scott), as a set of *points of reference*. $\mathcal{R}$ is the relation of relative possibility, or accessibility, among possible worlds, '$\alpha\mathcal{R}\beta$' reads 'β is possible with respect to α'. The function $\mathcal{D}$ determines the domain of individuals existing in any given possible situation; where $\alpha\in\mathcal{K}'$, $\mathcal{D}_\alpha$ is to be the range in situation α of bound individual variables. $\mathcal{D}'$ is the *outer domain*: a set of "individuals" which exist in no possible world; this is a device to allow for the arbitrary assignment of truth-values to formulas containing non-referring terms. The set D is defined as the union of all domains, including the outer domain: $D=\mathcal{D}'\cup\bigcup_{\alpha\in\mathcal{K}'}\mathcal{D}_\alpha$.

In modal logic, formulas are constructed using a modal operator, say $\Box$; a formula $\Box A$ is said to be true in a situation α in case A

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[3] We follow Church's conventions for abbreviating names of wffs by eliminating parentheses and using dots; see [3], pp. 74—80. The conditional connective $>$ is placed with the horseshoe in the highest category.

[4] These definitions are justifiable semantically; given our interpretation of the conditional connective, $\sim A>A$ and $\sim(A>\sim A)$ behave semantically just as $\Box A$ and $\Diamond A$ do in Kripke's semantics for modal logic.

is true in all β such that $\alpha\mathcal{R}\beta$. Under this understanding of things, a model structure $\langle\mathcal{K}', \mathcal{R}, \mathcal{D}, \mathcal{D}'\rangle$, where $\mathcal{K}'$, $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{D}'$ are as in *D3.1*, will be a model structure for the system Q3M of modal logic discussed in [18] and [19].[5] Here, 'Q3' stands for the interpretation of the quantifiers; in particular, the domain of quantification is allowed to vary from situation to situation. And 'M' stands for von Wright's minimal system of modal logic. As Kripke has shown, other familiar systems of modal logic can be characterized by imposing certain conditions on the relation $\mathcal{R}$.

The chief semantic difference between the system CQ of conditional logic and Q3M lies in the interpretation of the conditional connective; the semantic rule for > is that a formula $A>B$ is true in a situation α in case B is true in some selected situation possible with respect to α in which A is true. The component λ of a QMms is the *absurd world*—a situation in which all formulas are "true". This addition to the model structure is a device enabling us to assign a truth-value to $A>B$ where A is necessarily false.[6] In the following definitions, the notion of a "selected situation in which A is true" is made precise.

D3.2. A *sequence* σ on a morphology M and QMms $\langle\mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}'\rangle$ is a function taking members of V_M (individual variables) x into members $\sigma(x)$ of D.

By 'σ^d/x', we shall understand the sequence which differs from σ only in assigning d to x.

D3.3. A *valuation* of a morphology M on a QMms $\langle\mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}'\rangle$ is a function v assigning, for each member α of $\mathcal{K}'$, (*i*) a value $v_\alpha(P)$ in {T, F} to each 0-ary predicate letter P of M; (*ii*) a subset $v_\alpha(Q)$ of the cartesian product D^n to each n-ary predicate letter Q of M; (*iii*) to each individual constant a of M a member $v_\alpha(a)$ of D.

D3.4. An *s-function* on a QMms $\mathcal{M} = \langle\mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}'\rangle$ and morphology M is a function f which assigns to each wff A, each $\alpha \in \mathcal{K}'$, and each sequence σ on M and $\mathcal{M}$ a member $f(A, \alpha, \sigma)$ of $\mathcal{K}'$ meeting the following condition: for all A, α, and σ, if $f(A, \alpha, \sigma) \neq \lambda$, then $\alpha\mathcal{R}f(A, \alpha, \sigma)$.

The intuitive idea behind our account of conditionals is that a conditional statement asserts that its consequent is true in a particular situation determined by its antecedent. According to this idea, a determinate interpretation of a language involving conditional statements will require a selection rule picking a possible world for each antecedent expressible in the language. It is natural to make this rule depend also on the situation in which the choice is made, and in order to interpret formulas such as $(x)(P(x)>Q(x))$ it is necessary to make it also depend on sequences. Such a rule of selection is represented in our semantic account by an s-function. Our theory requires that in every interpretation of the formal system a particular s-function must be specified; in natural languages, however, it is likely that the selection rule is only partially specified by the conventions of the language, since it is easy to construct conditional statements which are context-dependent, ambiguous, or completely indeterminate.[7]

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[5] This holds with one exception; the domains $\mathcal{D}_\alpha$ are required to be nonempty in [19]; here, they are not. The axiomatic changes made necessary by this difference are minor.

[6] We do not regard the absurd world as a world, any more than imitation leather is leather; it is merely a device for handling the truth-values of conditionals with impossible antecedents. There are many equivalent ways of doing this: for instance, we could make f a partial function into $\mathcal{K}$ and say that $I_\alpha(A) = T$ if $I_\beta(A) = T$ for all β such that $\beta = f(\alpha)$.

[7] We are thinking, of course, of examples of the sort raised by Goodman [4], Rescher [12], and others. Our strategem here of making semantically determinate an assumption which often is not specified exactly in ordinary situations is a commonplace logical technique. For example, in the usual semantics of the functional calculus of first order, the truth-value of a quantified formula is undetermined unless a domain is given for the quantified variable, although such domains are not often completely specified in ordinary discourse.

In a model in which any semantic component, such as domain or an s-function, is not fully specified, some formulas will be interpreted as neither true nor false. Since there certainly exist grammatical sentences of natural languages which fail to make statements, a more realistic logical theory would allow for such truth-value gaps. Particularly relevant to conditional logic are gaps arising from conditions such as the antecedent of 'If Bizet and Verdi were compatriots then Bizet would be Italian', which do not suffice to pick out a single situation.

In the study of conditionals we are interested only in selection rules which meet certain formal requirements (e.g. the antecedent must be true in the situation selected, and the actual situation must be selected if the antecedent is in fact true in the actual situation). We shall take these constraints into account in our definition of an *interpretation*, but in order to formulate them we shall first define satisfaction for the more general notion of a *quasi-interpretation*.

D3.5. A *quasi-interpretation* I of a morphology **M** on a QMms $\mathcal{M}$ is an ordered pair $\langle v, f\rangle$, where v is a valuation of **M** on $\mathcal{M}$ and f is an s-function on $\mathcal{M}$ and **M**.

D3.6. Let $I=\langle v, f\rangle$ be a quasi-interpretation of **M** on a CQms $\mathcal{M}=\langle \mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}'\rangle$. Then for all $\alpha \in \mathcal{K}'$:

1) $I_\alpha(P, \sigma)=T$ iff $v_\alpha(P)=T$
2) $I_\alpha(a, \sigma)=v_\alpha(a)$
3) $I_\alpha(x, \sigma)=\sigma(x)$
4) $I_\alpha(P(t_1, \ldots, t_n), \sigma)=T$ iff $\langle I_\alpha(t_1, \sigma), \ldots, I_\alpha(t_n, \sigma)\rangle \in v_\alpha(P)$
5) $I_\alpha(s=t, \sigma)=T$ iff $I_\alpha(s, \sigma)=I_\alpha(t, \sigma)$
6) $I_\alpha(\sim A, \sigma)=T$ iff $I_\alpha(A, \sigma)=F$
7) $I_\alpha(A \supset B, \sigma)=T$ iff $I_\alpha(A, \sigma)=F$ or $I_\alpha(B, \sigma)=T$
8) $I_\alpha(A>B, \sigma)=T$ iff $I_{f(A, \alpha, \sigma)}(B, \sigma)=T$
9) $I_\alpha((x)A, \sigma)=T$ iff for all $d \in \mathcal{D}_\alpha$, $I_\alpha(A, \sigma^d/x)=T$
10) $I_\lambda(A, \sigma)=T$ for all A and σ.

For all formulas A, if $I_\alpha(A, \sigma)\neq T$, then $I_\alpha(A, \sigma)=F$.

The value $I_\alpha(A, \sigma)$ defined here is the truth-value given by I to A in a situation α, relative to an assignment σ of values to individual variables.

However, we feel that the best way to deal with such problems is first to construct a two-valued theory, which can be regarded as an idealization in which semantic information is completely specified. This theory can then be modified to allow for incomplete s-functions. Here we have dealt only with the first part of this program. See van Fraassen [21] for a very elegant and easily generalized approach to the second.

D3.7. An *interpretation* of **M** on a QMms $\mathcal{M}=\langle \mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}'\rangle$ is a quasi-interpretation $I=\langle v, f\rangle$ of **M** on $\mathcal{M}$ which for all wffs A and B of **M**, all sequences σ and τ on **M** and $\mathcal{M}$, and all $\alpha \in \mathcal{K}'$ meets the following four conditions:

1) $I_{f(A, \alpha, \sigma)}(A, \sigma)=T$;
2) If $I_\alpha(A, \sigma)=T$, then $f(A, \alpha, \sigma)=\alpha$;
3) $f(A, \alpha, \sigma)=\lambda$ only if there is no $\beta \in \mathcal{K}'$ such that $\alpha\mathcal{R}\beta$ and $I_\alpha(A, \sigma)=T$;
4) If $I_{f(A, \alpha, \sigma)}(B, \tau)=I_{f(B, \alpha, \tau)}(A, \sigma)=T$, then $f(A, \alpha, \sigma)=f(B, \alpha, \tau)$.

Definition *D3.7* states the constraints relevant to selection rules for conditionals. Condition (1) requires that the antecedent be true in the world selected; to say something of the form 'If *P*, then ...' is to imagine a situation in which *P* is true. Condition (2) requires that the actual situation be selected if it is eligible (that is, if it meets condition (1)): if one asserts a conditional, one is committed to its consequent, should the antecedent be true. Condition (3) ensures that the absurd world can be chosen only when the antecedent is impossible: to suppose something which in fact is possible is to imagine a possible situation. Condition (4) ensures that possible situations are selected in a uniform fashion, in the following sense: if a situation α is chosen over β in one context *in which both are eligible* (i.e. with respect to an antecedent true in both), then α must be chosen over β in all contexts in which both are eligible.

Intuitively, an s-function represents a rule involving inductive and theoretical priorities among situations. An s-function that is a component of an interpretation meeting the above four conditions is a function that determines a simple ordering of certain of the situations possible with respect to each situation α; this ordering should be thought of as an order of *resemblance* to α. The function then selects from among the situations in which the antecedent of a conditional is true the one that is most like the actual situation. How this order is determined in practice is an empirical and methodological question rather than a formal one. Our constraints require only that the ordering be simple, that the actual situation be prior to all others, and that all possible situations be prior to the absurd world.

D3.8. A set Γ of wffs of **M** is *simultaneously satisfiable* if there exists an interpretation I of **M** on a CQms $\langle \mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}' \rangle$, a sequence σ on $\langle \mathcal{K}, \mathcal{R}, \lambda, \mathcal{D}, \mathcal{D}' \rangle$ and a member α of $\mathcal{K}'$ such that for all $A \in \Gamma$, $I_\alpha(A, \sigma) = T$.

D3.9. A wff A is *valid* if $\{\sim A\}$ is not simultaneously satisfiable.

4. *Deducibility*

In this section we shall define a formal system CQ by setting down axiom schemata and rules of inference sufficient to characterize syntactic notions of consistency and theoremhood corresponding to the concepts of simultaneous satisfiability and validity defined above. We shall list without proof some object-language theorem schemata and some syntactic metatheorems to be used in the next section.

D4.1. A wff of a morphology **M** is an *axiom* of CQ if it is tautologous or has one of the following forms:

A1. $\Box(A \supset B) \supset . \Box A \supset \Box B$
A2. $\Box(A \supset B) \supset . A > B$
A3. $\Diamond A \supset . A > B \supset \sim(A > \sim B)$
A4. $A > (B \vee C) \supset . (A > B) \vee (A > C)$
A5. $A > B \supset . A \supset B$
A6. $(A > B) \wedge (B > A) \supset . (A > C) \supset (B > C)$
A7. $(x)A \supset . E\Box t \supset A^t/x$
A8. $(x)(E\Box x \supset A) \supset (x)A$
A9. $s = s$
A10. $s = t \supset . A \supset A^s//t$, where no occurrence of t in A that is replaced by s falls within the scope of a modal operator
A11. $\Diamond x = y \supset \Box x = y$

D4.2. A wff of a morphology **M** is a *theorem* of CQ if it is an axiom, or if it follows from theorems by any of the following five rules.

R1. If $A \supset B$ and A are theorems, then B is a theorem.
R2. If A is a theorem, then $B > A$ is a theorem.
R3. If $A \supset B$ is a theorem and x has no free occurrences in A, then $A \supset (x)B$ is a theorem.
R4. If $A_1 > . A_2 > \ldots > . A_n > B$ is a theorem and x has no free occurrences in $A_1, A_2, \ldots,$ or A_n, then $A_1 > . A_2 > \ldots > . A_n > (x)B$ is a theorem.
R5. If $A_1 > . A_2 > . \ldots > . A_n > \sim t = x$ is a theorem and x has no free occurrences in $A_1, A_2, \ldots,$ or A_n, then $A_1 > . A_2 > . \ldots > . \sim A_n$ is a theorem.

In the usual way, these axioms and rules determine syntactic notions of CQ-derivability and CQ-consistency; '$\vdash A$' and '$\Gamma \vdash A$' shall be understood to mean that A is a theorem of CQ and that A is derivable from Γ in CQ, respectively.[8]

We list here without proof some theorem schemata and derived rules of CQ.

Theorem schemata:

t4.3. $\vdash \Box A \supset A$
t4.4. $\vdash A > A$
t4.5. $\vdash A > \sim A \supset . A > B$
t4.6. $\vdash \sim A > A \supset . B > A$
t4.7. $\vdash (A > B) \vee (A > \sim B)$
t4.8. $\vdash (A_1 > . A_2 > . \ldots > . A_n > B) \vee (A_1 > . A_2 > . \ldots > . A_n > \sim B)$
t4.9. $\vdash A \supset . B \equiv (A > B)$
t4.10. $\vdash x = y \supset . A \equiv A^y/x$
t4.11. $\vdash (x)B \supset . Ey \supset B^y/x$

[8] Specifically, an occurrence B_i of a formula in a sequence $B_1, \ldots, B_n$ of formulas is *categorical* if some subsequence of $B_1, \ldots, B_n$ is a proof in CQ of B_i. A sequence $B_1, \ldots, B_n$ is a *deduction* in CQ of B_n from a set Γ of formulas if for all i such that $1 \leq i \leq n$, either 1) B_i is categorical in $B_1, \ldots, B_n$ or 2) $B_i \in \Gamma$, or 3) for some $j, k < i$, B_j is $B_k \supset B_i$.

Derived rules:

T4.12. If $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \supset B$.

T4.13. If $\Gamma \vdash A_1 >.\ A_2 >.\ \ldots >.\ A_n > B_i$ for all i, $1 < i < m$, and $\{B_1, \ldots, B_m\} \vdash C$, then $\Gamma \vdash A_1 >.$ $>.\ A_2 >.\ \ldots >.\ A_n > C$.

T4.14. If $\Gamma \vdash \sim(A_1 >.\ A_2 >.\ \ldots >.\ A_n > \sim A_n)$, then $\Gamma \vdash A_1 >.\ A_2 >.\ \ldots >.\ A_n > x = y$ iff $\Gamma \vdash x = y$.

5. *Saturated sets*

In the proof we will give in section 6 of completeness, the notion of an M-*saturated set* plays a crucial role. Intuitively, an M-saturated set is a set of formulas of M having syntactic properties ensuring that it is a set of truths in some situation, relative to an assignment of values to variables which gives every individual a name. In this section we will define this notion and state some of its more important properties.

D5.1. A set Γ of wffs of a morphology M is M-*saturated* if it meets the following four conditions:

1) Γ is CQ-consistent;
2) For all $A \in W_M$, either $A \in \Gamma$ or $\sim A \in \Gamma$;
3) For all $n \geq 0$, all $A_1, \ldots, A_n, B \in W_M$, and all $x \in V_M$, there is a $y \in V_M$ such that $A_1 >.\ \ldots >.\ A_n > (B^y/x \supset (x)B) \in \Gamma$;
4) For all $n > 1$, all $A_1, \ldots, A_n \in W_M$ and $t \in T_M$, there is a $y \in V_M$ such that $A_1 >.\ \ldots >.\ A_n > t = y \in \Gamma$

D5.2. An ω-*extension* of a morphology M is a morphology M' like M except that $V_{M'} = V_M \cup \{x_1, x_2, \ldots\}$, where $x_1, x_2, \ldots$ are symbols foreign to M.

L5.3. Let M be a morphology, and M' be any ω-extension of M. Then every consistent set of wffs of M has an M'-saturated extension.

This lemma is proved in the same way as its modal analogue,

using R4 and R5[9]. And the following lemma is an immediate consequence of *D5.1*.

L5.4. If Γ is M-saturated, then (*i*) for all wffs A and individual variables x of M, $(x)A \in \Gamma$ if $A^y/x \in \Gamma$ for all y such that $y \in V_M$ and $Ey \in \Gamma$; and (*ii*) for all $t \in T_M$ there is a $y \in V_M$ such that $t = y \in \Gamma$.

L5.5 For every $A_1, A_2, \ldots, A_n \in W_M$ and subset Γ of W_M, if Γ is M-saturated and $\Delta = \{B : (A_1 >.\ A_2 >.\ \ldots >.\ A_n > B) \in \Gamma\}$, then either $\Delta = W_M$ or Δ is M-saturated.

PROOF. Δ is either inconsistent or consistent. We shall show that $\Delta = W_M$ if inconsistent, and Δ is M-saturated if consistent. Suppose it is inconsistent; then there is a finite set $\{B_1, \ldots, B_m\} \subseteq \Delta$ such that $\{B_1, \ldots, B_m\} \vdash C$ for all $C \in W_M$. Then by *T4.13*, $(A_1 >.\ A_2 >.$ $>.\ \ldots >.\ A_n > C) \in \Gamma$ for all $C \in W_M$, so $\Delta = W_M$. On the other hand, suppose Δ is consistent; then it meets condition 1 of *D5.1*. Since Γ is M-saturated, and in view of *t4.8*, either $B \in \Delta$ or $\sim B \in \Delta$, so condition 2 is satisfied. As for condition 3, let $B_1, B_2, \ldots, B_m$, and B be any $m+1$ wffs of M. Since Γ is M-saturated, for all $x \in V_M$ there is a $y \in V_M$ such that $(A_1 >.\ A_2 >.\ \ldots >.\ A_n >.\ B_1 >$ $>.\ \ldots >.\ B_m > (B^y/x \supset (x)B)) \in \Gamma$. Therefore, $(B_1 >.\ B_2 >.\ \ldots >$ $>.\ B_m > (B^y/x \supset (x)B)) \in \Delta$, by definition. A similar argument shows that Δ meets condition 4 of *D.5.1*.

6. *Semantic completeness*

It is readily verified that all of the axioms listed in *D4.1* are valid, and that the rules listed in *D4.2* preserve validity. Consequently, we have at once the following results.

T6.1. Every theorem of CQ is valid.

T6.2. For any set Γ of wffs, if Γ is simultaneously satisfiable then Γ is CQ-consistent.

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9 See [19], lemma 11.

We will use the familiar method due to Henkin to prove the converses of these theorems. Given an arbitrary consistent set Γ of wffs of M, we will construct a CQms using certain M'-saturated sets as the members of $\mathcal{K}'$; then we will define an interpretation on the model structure which simultaneously satisfies Γ in some member of $\mathcal{K}'$.

To characterize this model structure, we need the following preliminary definitions. Let M' be an ω-extension of the morphology M, and Γ be an M'-saturated set. For all $x, y \in V_M$, let $x \sim y$ iff $x = y \in \Gamma$; this is an equivalence relation, and so splits V_M into equivalence classes. Let g be a function selecting one member of each equivalence class, so that $x \sim y$ iff $g(x) = g(y)$.

D6.3. Let $\mathcal{M} = \langle \mathcal{K}, \mathcal{R}, W_{M'}, \mathcal{D}, \mathcal{D}' \rangle$, where $\mathcal{K}$ $(= \mathcal{K}(\Gamma))$ is defined as follows: $\Delta \in \mathcal{K}$ iff there is a finite set $\{A_1, A_2, \ldots, A_n\} \subseteq W_{M'}$ such that $\{B: (A_1 >. A_2 >_2 \ldots >. A_n > B) \in \Gamma\} = \Delta$. $\mathcal{K}' = \mathcal{K} - \{W_{M'}\}$; $\mathcal{R}$ is defined on $\mathcal{K}'$ so that $\Delta_1 \mathcal{R} \Delta_2$ iff for some $A \in W_{M'}$, $\Delta_2 = \{B : A > B \in \Delta_1\}$; $\mathcal{D}_\Delta = \{g(x) : Ex \in \Delta\}$; and $\mathcal{D}' = \{g(x) :$ for all $\Theta \in \mathcal{K}'$, $Eg(x) \notin \Theta\}$. Comparing *D3.1* with *D6.3*, it can be seen that $\mathcal{M}$ is a CQms (it follows at once from *t4.3* that $\mathcal{R}$ is reflexive).

Before defining an appropriate valuation function v on $\mathcal{M}$, we prove the following presupposition of this definition.

L6.4. For all $c \in C_{M'}$ and $\Theta \in \mathcal{K}'$, there exists a unique $x \in D$ such that $x = c \in \Theta$.

PROOF. Since Θ is M'-saturated, by *L5.4* there is an $x \in V_{M'}$ such that $c = x \in \Theta$. By *T4.14* and the construction of $\mathcal{K}'$, for all $x, y \in V_{M'}$, $x = y \in \Theta$ iff $x = y \in \Gamma$; in other words, $x = y \in \Theta$ iff $x \sim y$. By this and the properties of identity, it follows that $\{y : y = c \in \Theta\}$ is one of the equivalence classes under $\sim$. But only one member of each equivalence class is a member of D.

D6.5. Let v be defined as follows: for all $\Theta \in \mathcal{K}'$, (*i*) for each 0-ary predicate letter P of M', $v_\Theta(P) = T$ if $P \in \Theta$, and $v_\Theta(P) = F$ otherwise; (*ii*) for each *n*-ary predicate letter Q, $v_\Theta(Q) = \{\langle g(x_1), \ldots, g(x_n) \rangle : Q(x_1, \ldots, x_n) \in \Theta\}$; (*iii*) for each $c \in C_{M'}$, $v_\Theta(c)$ is the unique $x \in D$ such that $x = c \in \Theta$. Comparing *D3.5* and *D6.5*, it can be seen that v is a valuation of M' on $\mathcal{M}$.

Before defining an appropriate s-function on $\mathcal{M}$, we will point out two relevant facts about sequences. First, since the domain of $\mathcal{M}$ is itself a set of individual variables of M' a sequence on M and $\mathcal{M}$ may be used as a substitution function for wffs. We shall use 'A^σ' to signify the wff resulting from replacing, in alphabetical order, the variables x having free occurrences in A by $\sigma(x)$, relettering bound variables if necessary. Second, note that the function g used in defining $\mathcal{M}$ is itself a sequence on M and $\mathcal{M}$. We will define our s-function f first for this basic sequence, and then for sequences in general.

D6.6. Let f be a function defined as follows: for all $A \in W_{M'}$ and $\Theta \in \mathcal{K}'$, $f(A, \Theta, g) = \{B : A > B \in \Theta\}$, and for all sequences σ on M and $\mathcal{M}$, $f(A, \Theta, \sigma) = f(A^\sigma, \Theta, g)$.

This definition presupposes that $f(A^g, \Theta, g) = f(A, \Theta, g)$, for all $A \in W_{M'}$ and $\Theta \in \mathcal{K}'$. But this is easily established, using *t4.10*.

Comparing *D3.4* and *D6.6* in the light of *D6.3* and *L5.5*, it is easily seen that f is an s-function on M and $\mathcal{M}$. Therefore, $I = \langle v, f \rangle$ is a quasi-interpretation of M on $\mathcal{M}$.

To prove our final two lemmas, we will need the following relation between semantic and syntactic substitution, which is easily established by induction on the complexity of A.

L6.7. For all $A \in W_{M'}$, $\Theta \in \mathcal{K}'(\Gamma)$ and sequences σ on M and $\mathcal{M}$, $I_\Theta(A, \sigma) = I_\Theta(A^\sigma, g)$.

L6.8. For all $A \in W_{M'}$, $I_\Theta(A, g) = T$ iff $A \in \Theta$.

PROOF. First, by clause (10) of *D3.6*, $I_{W_{M'}}(A, g) = T$ for all $A \in W_{M'}$. Therefore $I_{W_{M'}}(A, g) = T$ iff $A \in W_{M'}$. For any $\Theta \in \mathcal{K}'$, the lemma is proved by induction on the complexity of A; we will present here only the cases involving the corner and the quantifier. First, suppose A has the form $B > C$. Then $I_\Theta(A, g) = T$ iff $I_{f(B, \Theta, g)}(C, g) = T$. By the hypothesis of induction, this iff $C \in f(B, \Theta, g)$, and by *D6.6*, this iff $A \in \Theta$. Second, suppose A has the form $(x)B$. Then $I_\Theta(A, g) = T$ iff $I_\Theta(B, g^{g(y)}/x) = T$ for all y such that $Ey \in \Theta$. By *L6.7*, this iff $I_\Theta(B^{g(y)}/x, g) = T$ for all y such that $Ey \in \Theta$, and by the hypothesis of induction, this iff $B^{g(y)}/x \in \Theta$ for all y such that $Ey \in \Theta$. Now in view of *t4.10*, for all $y \in V_{M'}$, $B^{g(y)}/x \in \Theta$ iff $B^y/x \in \Theta$. So $B^{g(y)}/x \in \Theta$ for all y such that $Ey \in \Theta$ iff $B^y/x \in \Theta$ for all y such that $Ey \in \Theta$. And by *L5.4*, if $B^y/x \in \Theta$ for all $y \in V_{M'}$ such that $Ey \in \Theta$, then $(x)B \in \Theta$. Conversely, if $(x)B \in \Theta$, then $Ey \supset B^y/x \in \Theta$ for all $y \in V_{M'}$, by *t4.11*. Hence $B^y/x \in \Theta$ for any y such that $Ey \in \Theta$. Therefore $I_\Theta(A, g) = T$ iff $A \in \Theta$, when A has the form $(x)B$.

L6.9. $I = \langle v, f \rangle$ is an interpretation of **M** on $\mathcal{M}$.

PROOF. Following *D3.7*, we must show that for all $A, B \in W_{M'}$, all $\Theta \in \mathcal{K}'(\Gamma)$ and all sequences σ and τ on $\mathcal{M}$, (1) $I_{f(A,\Theta,\sigma)}(A, \sigma) = T$; (2) if $I_\Theta(A, \sigma) = T$, then $f(A, \Theta, \sigma) = \Theta$; (3) $f(A, \Theta, \sigma) = W_{M'}$ only if there is no $\Xi \in \mathcal{K}'(\Gamma)$ such that $I_\Xi(A, \sigma) = T$ and $\Theta \mathcal{R} \Xi$; (4) if $I_{f(A,\Theta,\sigma)}(B, \tau) = I_{f(B,\Theta,\tau)}(A, \sigma) = T$, then $f(A, \Theta, \sigma) = f(B, \Theta, \tau)$. For notational convenience in proving these four, let A' be A^σ and B' be B^τ. (1) By *L6.7* and *D6.6*, $I_{f(A,\Theta,\sigma)}(A, \sigma) = I_{f(A',\Theta,g)}(A', g)$. By *t4.4*, $A' > A' \in \Delta$, so that $A' \in f(A', \Theta, g)$, and by *L6.8*, $I_{f(A',\Theta,g)}(A', g) = T$. Thus, $I_{f(A,\Theta,\sigma)}(A, \sigma) = T$. (2) Assume $I_\Theta(A, \sigma) = I_\Theta(A', g) = T$. Then $A' \in \Theta$, so by *t4.9*, $B \equiv (A' > B) \in \Theta$ for all $B \in W_{M'}$, so $\{B : A' > B \in \Theta\} = \Theta$, and therefore $f(A, \Theta, \sigma) = f(A', \Theta, g) = \Theta$. (3) Suppose that $f(A, \Theta, \sigma) = f(A', \Theta, g) = W_{M'}$ and suppose also that $\Theta \mathcal{R} \Xi$. For *reductio*, assume that $A' \in \Xi$. By the definition of $\mathcal{R}$ in *D6.3*, since $\Theta \mathcal{R} \Xi$, $\sim(A' > \sim A') \in \Theta$. But since $f(A', \Theta, g) = W_{M'}$, $A' > \sim A' \in \Theta$. So since Θ is consistent, the assumption is false. Thus, for all Ξ such that $\Theta \mathcal{R} \Xi$, $A' \notin \Xi$; by *L6.8*, for all Ξ such that $\Theta \mathcal{R} \Xi$, $I_\Xi(A, \sigma) = F$. (4) Let $f(A, \Theta, \sigma) = \Xi_1$, and let $f(B, \Theta, \tau) = \Xi_2$; suppose $I_{\Xi_1}(B, \tau) = I_{\Xi_2}(A, \sigma) = T$. Then $\Xi_1 = f(A', \Theta, g)$ and $\Xi_2 = f(B', \Theta, g)$; and $I_{\Xi_1}(B', g) = I_{\Xi_2}(A', g) = T$. Therefore $A' > B' \in \Theta$ and $B' > A' \in \Theta$, by *D6.6*. Since Θ is **M**'-saturated, it follows from A6 that for all wffs C of **M**', $A' > C \in \Theta$ iff $B' > C \in \Theta$, so $C \in \Xi_1$ iff $C \in \Xi_2$, for all $C \in W_{M'}$. Therefore $\Xi_1 = \Xi_2$. Hence, by (1)—(4), I is an interpretation.

We have now shown that the model structure $\mathcal{M}$ of *D6.3* is a **CQ**ms and that the quasi-interpretation I of *D6.5* and *D6.6* is an interpretation. By *L6.8*, $I_\Gamma(A, g) = T$ if $A \in \Gamma$; the set Γ is therefore simultaneously satisfiable. But all that was assumed concerning Γ was that it is **M**'-saturated. We have therefore established the following lemma.

L6.10. Any **M**'-saturated set is simultaneously satisfiable.

By *L5.3*, any consistent set Γ of wffs of **M** has an **M**'-saturated extension Δ, where **M**' is any ω-extension of **M**. Combining this with *L6.10*, we obtain the desired completeness theorem.

T6.11. Any consistent set of wffs of **M** is simultaneously satisfiable.

Combining this with *T6.2*, we obtain the equivalence of **CQ**-consistency and satisfiability; finally, weak completeness follows as a corollary.

T6.12. Any set of wffs is **CQ**-consistent iff it is simultaneously satisfiable.

T6.13. Any wff is a theorem of **CQ** iff it is valid.

7. *Extensions and modifications*

The familiar connectives of modal logic are definable within the system **CQ** of conditional logic, and **CQ** can be adjusted and embellished in much the same ways as systems of modal logic. In particular, conditional logic can be made to conform to various theories of individuals. Definite descriptions can be added to the language of **CQ**, as can epistemic, deontic, and other modal operators.

We have chosen von Wright's **M** as the underlying modal logic of CQ because it is a minimal theory of necessity.[10] To obtain a system based on stronger theories such as S4 or S5, one need only place additional constraints on the relation $\mathcal{R}$ (for S4, $\mathcal{R}$ must be transitive; for S5, transitive and symmetric), and add the characteristic axioms $\Box A \supset \Box\Box A$ for S4 and $\Diamond A \supset \Box\Diamond A$ for S5. In CQ, the quantifier ranges over *actual* individuals—those individuals existing in the possible world under consideration. This means that the domains must be allowed to vary from situation to situation. In modal logic, the variability of domains renders the system and the completeness proof considerably more complicated, and partly for this reason many writers have used a single domain (understood intuitively as the set of all possible individuals) and a quantifier (called the outer quantifier) ranging over this domain. The inner quantifier, ranging over only actual individuals, may then be defined in terms of the outer quantifier and a primitive predicate of existence.

But in conditional logic many of the technical problems with variable domains disappear, and the proof of completeness that we have given is as simple as the comparable proof for the system with outer quantifiers and a single domain. Those who nevertheless prefer quantification over possible individuals may satisfy themselves by adding to our semantics a requirement that $\mathcal{D}_\alpha = \mathcal{D}_\beta$ for all $\alpha, \beta \in \mathcal{K}'$, and perhaps a requirement that the domain be nonempty. A simpler axiomatization is then possible.

Formulas containing definite descriptions may be added to a morphology by allowing complex terms of the sort $I_x A$, where x is any individual variable and A any formula of the morphology. The term $I_x A$ denotes in a member α of $\mathcal{K}'$ the unique individual existing in α which satisfies A in α, if there is such an individual; and otherwise it denotes some arbitrary nonexisting individual (i.e., a member of $D - \mathcal{D}_\alpha$).

Formulas containing abstracts are added to **M** by allowing formulas $\hat{x}A(t)$, where $x \in V_M$, $A \in W_M$, and $t \in T_M$. The semantic rule for such formulas is that $I_\alpha(\hat{x}A(t), \sigma) = T$ iff $I_\alpha(t, \sigma) \in \{d: d \in D$

[10] More precisely, **M** is the system characterized by those model structures in which the only constraint on the relation $\mathcal{R}$ is that it be reflexive.

and $I_\alpha(A, \sigma^d/x) = T\}$. In classical logic, such formulas are superfluous in virtue of the abstraction principle: $\hat{x}A(t) \equiv A^t/x$. But in modal and conditional logic, the principle fails; for instance, $\hat{x}(A > B)(t)$ neither implies nor is implied by $(A > B)^t/x$.[11]

Epistemic, deontic and other modal operators can of course be added to conditional logic, and we believe their interaction with conditionals is of philosophical interest. For example, most of the problems in the literature concerning conditional obligation can be handled simply by combining a standard system of deontic logic with conditional logic: the deontic conditional then has the form $A > OB$. Also, we suspect that an analysis of *prima facie* obligation can be carried out within this framework.

8. *Philosophical applications*

The most important test of the theory we have presented will come in its application to philosophical problems. To be a good account of conditionality, it should help to clarify some of the many puzzles that have arisen in this area. The beginnings of this philosophical work have appeared in Stalnaker [14] and [15]; further applications and refinements will, we hope, appear later. In the present paper, we will limit ourselves to a few brief remarks indicating the direction we expect this work to take.

The theory suggests an account of the notion of a *law of nature*, and of the idea of nomological connection which underlies this notion. A law of nature has usually been thought of as a universal conditional statement. For certain purposes, however, it is necessary to distinguish genuine law-like statements from so-called "accidental" generalizations—universal conditionals that just happen to be true. This distinction has proved difficult to draw.

According to the analysis suggested by our theory of conditionals, a simple law-like statement has the form $(x)(P(x) > Q(x))$. It may be contrasted with the weaker statement, $(x)(P(x) \supset Q(x))$, which gives the form of an accidental generalization. According to this proposal, a law-like statement actually makes a stronger

[11] For more information on this topic, see [17] and [20].

claim, and has different truth conditions than the accidental universal conditional. However, the difference in the two generalizations can only be brought out within an intensional language, since they make exactly the same claim about the actual world. Both statements assert that all actual P's are Q's, but the law-like statement adds the *counterfactual* claim that all non-P's *would* be Q's if they were P's.

Although this explication is given within a modal language, it is quite different from an anlysis in terms of physical or empirical necessity. On our view, nomological connection is not a kind of necessity; the logic of laws is very different from the logic of strict implication. Because of this difference in structure, our analysis (1) provides for a novel approach to the paradoxes of confirmation, and (2) allows for an interesting connection with inductive probability.

(1) If law-like connection is represented by the conditional corner, then the paradoxes concerning contraposition do not arise at all, since $(x)(P(x)>Q(x))$ neither implies nor is implied by $(x)(\sim Q(x)> \sim P(x))$. Furthermore, the semantic theory enables one to see why a conditional might differ in confirmation and in truth value from its contrapositive.

Suppose that all ravens are black. For the moment, represent this statement by the universal material conditional, $(x)(R(x) \supset B(x))$. Assume also that the class of ravens mentioned in the antecedent is a *natural kind*, or that the property of being a raven is a *substance property*, or a *sortal universal*. This entails that the ravenhood of a raven is a criterion of its identity—that to remain the *same* thing, a raven must remain a raven. Formally, this means that $(x)(\Diamond R(x) \supset \Box R(x))$ is true.

Now from these two assumptions as premisses, the *law-like* statement, "All ravens are black," follows in CQ; that is, $\{(x)(R(x) \supset B(x)),\ (x)(\Diamond R(x) \supset \Box R(x))\} \models (x)(R(x)>B(x))$. Thus, a universal conditional statement whose antecedent predicate expresses a natural kind will always determine a law-like statement. Consider, however, the contrapositive, $(x)(\sim B(x) \supset \sim R(x))$. Here, the antecedent predicate picks out quite an unnatural class, the class of non-black things. Hence the law-like analogue is not plausible, even when the universal material conditional is true.

(2) It has been suggested that one might characterize conditionals and laws of nature by defining their *probability values* in terms of the probabilities of their constituents rather than defining their truth-values in terms of their constituent truth-values. If this were done, the following definition seems natural: $\Pr(A>B) = \Pr(B/A)$; the probability of a conditional is the same as the conditional probability of the consequent on the condition of the antecedent. If this definition is extended in a natural way, the resulting conditional concept has exactly the structure of the conditional defined by the system **CQ**. This relationship between the probability calculus and conditional logic provides some support for our analysis as well as an opportunity to draw a connection between inductive logic and conditional logic. This connection is explored, using propositional logic only, in [15]. Now that the system has been extended to include quantifiers, more detailed investigation of the relationships among counterfactuals, laws, and probabilities will be possible.

We have talked about laws of nature, but we might instead have talked directly about the relationships expressed by laws. One of the advantages of the semantic approach is that it allows us to focus directly on the world rather than on linguistic expressions purporting to describe it. One might, for example, give an analysis of explanation in terms of possible worlds and s-functions rather than indirectly in terms of laws. This way, one could avoid the embarrassment of unstated or unknown laws, and one might find an analysis which handles more plausibly the everyday cases of explanation.

These are of course suggestions, not final solutions. Before a logical theory is applied to philosophical material, it should be shown to be technically respectable; it should be articulated in a way satisfying the standards currently set by logicians. Our aim has been to accomplish this and to provide only some rough ideas about how the theory may be used to clarify philosophical problems.

BIBLIOGRAPHY

[1] Carnap, R., "Testability and meaning". *Philosophy of science*, vol. 3 (1936), pp. 420—468.

[2] Chisholm, R., "The contrary-to-fact conditional". *Mind*, n.s., vol. 55 (1946), pp. 289—302.

[3] Church, A., *Introduction to mathematical logic*, vol. 1, Princeton, 1956.

[4] Goodman, N., *Fact, fiction, and forecast*. Cambridge, Mass., 1955.

[5] Goodman, N., "The problem of counterfactual conditionals". *Journal of philosophy*, vol. 44 (1947), pp. 113—128. Reprinted as the first chapter of [4].

[6] Kripke, S., "Semantical analysis of modal logic I". *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 9 (1963), pp. 67—96.

[7] Lewis, C., *An analysis of knowledge and valuation*. La Salle, Ill., 1946.

[8] Nagel, E., *The structure of science*. New York, 1961.

[9] Pap, A., *An introduction to the philosophy of science*. Glencoe, Ill., 1962.

[10] Rescher, N., "Belief-contravening suppositions". *Philosophical review*, vol. 70 (1961), pp. 176—195.

[11] Rescher, N., "A factual analysis of counterfactual conditionals". *Philosophical studies*, vol. 11 (1960), pp. 49—54.

[12] Rescher, N., *Hypothetical reasoning*. Amsterdam, 1964.

[13] Ryle, G., "'If', 'so', and 'because'". *Philosophical analysis*, ed. M. Black, Ithaca, N.Y., 1950, pp. 323—340.

[14] Stalnaker, R., "A theory of conditionals". *Studies in logical theory* (*American philosophical quarterly* supplementary monograph series), Oxford, 1968.

[15] Stalnaker, R., "Probability and conditionals". *Philosophy of science*, vol. 37 (1970).

[16] Stalnaker, R., and R. Thomason, "A semantic analysis of conditional logic". Mimeographed, 1967.

[17] Stalnaker, R. and R. Thomason, "Abstraction in first order modal logic". *Theoria*, vol. 34 (1968), pp. 203—207.

[18] Thomason, R., "Modal logic and metaphysics". *The logical way of doing things*, ed. K. Lambert, New Haven, Conn., 1969.

[19] Thomason, R., "Some completeness results for modal predicate calculi". *Philosophical developments in non-standard logic*, ed. K. Lambert, forthcoming.

[20] Thomason, R., and R. Stalnaker, "Modality and reference". *Noûs*, vol. 2 (1968), pp. 359—372.

[21] van Fraassen, B., "Singular terms, truth-value gaps, and free logic". *Journal of philosophy*, vol. 63 (1966), pp. 481—495.

[22] von Wright, G., *Logical studies*. New York and London, 1957.

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